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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series makes available expository articles which appeared in a variety of mat matical periodicals. Topics covered include: (1) Helmholtz and the ture of geometrical axioms: (2) the straight line: (3) geometry and i tuition: and (4) the curvature of space. (MP)

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which do not find a place in the curriculum simply because of lack of time, even though they are well within the grasp of secondary school students.

Some classes and many individual students, however, may find time to pursue mathematical topics of special interest to them. The School Mathematics Study Group is preparing pamphlets designed to make material for such study readily accessible. Some of the pamphlets deal with material found in the regular curriculum but in a more extended manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum.

This particular series of pamphlets, the Reprint Series, makes available expository articles which appeared in a variety of mathematical periodicals. Even if the periodicals were available to all schools, there is convenience in having articles on one topic collected and reprinted as is done here.

This series was prepared for the Panel on Supplementary Publications by Professor William L. Schaaf. His judgment, background, bibliographic skills, and editorial efficiency were major factors in the design and successful completion of the pamphlets.

PREFACE

Geometry can mean different things to different people. It may evoke thoughts of design, drafting, architecture, art, measurement, mensuration, mechanics, engineering, surveying, geodesy, navigation, astronomy, and other equally significant activities. Or it may evoke thoughts of formal logic, deductive reasoning, propositions, definitions, postulationai systems, symbolic logic, and, in general, questions of mathematical philosophy and metamathematics.

In historical perspective, geometry was the first branch of mathematics to be cultivated systematically. From Thales to Apollonius, the Greeks erected an intellectual edifice that has withstood the impact of nearly 2000 years. This heritage from the ancients had been preceded by crude, empirical, working rules for measuring and carrying on such everyday activities as building and surveying. Indeed, the very word "geometry" means *earth-measure*. These two aspects of geometry suggest two antithetical approaches to the subject: the first, a purely logical, abstract point of view, where points and lines are thought of as "idealized constructs," and where the major emphasis is put upon hypothetico-deductive reasoning; the second, a frankly "practical," utilitarian point of view, where lines and angles are measured not for their own sake, but applied to physical objects as a means to some further end.

The past hundred years, however, have witnessed a remarkably changed outlook. To begin with, Euclidean geometry was "unshackled" by the creative imagination of Gauss, Bolyai, Lobachevski, and Riemann. Then, on the threshold of the twentieth century, geometry was identified with logic through the contributions of Peano, Hilbert, Veblen, Russell, Whitehead and others. Geometry was now regarded as a purely abstract, formal postulation system, employing "meaningless marks on paper" and virtually devoid of content; but it had a definite structure, and this was all-important. Subsequently, mathematicians have come to think of geometry, or more precisely, many geometries, not as systems of mathemaitcs *per se*, but rather as a particular way of looking at mathematica' ideas. In this sense geometry was found to be subtly and intimately associated with arithmetic, algebra, analysis, the theory of numbers, group theory, topology — indeed nearly every area of mathematics.

This collection of essays depicts the fascinating role played by man's intuition in the history of geometry. The human nervous system dictates certain conceptual images resulting from sensory experience and leading to certain notions such as *straight, curved, flat, round, parallel, continuous, finite, infinite, endless, boundary*, and the like. Contemporary mathematicians use these terms in a far more sophisticated way than suggested by ordinary usage.

—*William L. Schaaf*

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Morris Kline, *"The Straight Line,"* vol. 194 (March 1956), pp. 104-114.

P. LeCorbeiller, *"The Curvature of Space,"* vol. 191 (November 1954), pp. 80-86.

FOREWORD

As commonly used, the words "intuitive" and "intuition" have several meanings which, to some extent, overlap. We sometimes refer to a blind, uneducated guess as an intuitive inference, as, for example, that the earth is the center of the solar system, or that phlogiston is the combustible substance which is dispelled from an object upon heating it. Again, we sometimes speak of a shrewd intelligent guess as an intuitive conjecture: for example, that $x^n + y^n \neq z^n$ for $n > 2$, where x, y and z are any integers other than zero; or, that every even integer greater than 2 can be represented as the sum of two positive primes.

The commonest use of the word *intuitive*, however, is that suggested by such phrases as "common sense tells us that . . ." or "you just feel it in your bones that . . ."; the implication being that our sensory perception makes a very strong appeal to the intellect. For example, we readily assent to the equality of the measure of the opposite angles formed by two intersecting straight lines; we assert without hesitation that a straight line joining two given points is shorter than a polygonal path between them.

This latter use of the word *intuitive* is particularly pertinent to geometry. Many writers have pointed out that our concepts of geometric figures are derived basically from our sensory perception of objects in the external world. By manipulating, experimenting and observing physical entities, we arrive at certain generalizations concerning magnitude, form, position, and spatial relations.

This intuitive basis for our geometric concept is, however, only the beginning of a process of refinement which leads to thought about abstract entities and logical relations, i.e., to pure geometry. And, indeed, primitive generalizations based upon intuitive sense perception may exert a restrictive influence upon the creation of abstract mathematical ideas, as in the case of non-Euclidean geometry.

This first essay sketches the historical development leading to the mathematician's complete emancipation from the bonds of sensory experience in the realm of geometry.

Helmholtz and the nature of geometrical axioms: a segment in the history of mathematics

Morton R. Kenner

It was a hard struggle for mankind to free himself from the notion that geometry had anything to do with the real world.

"In short, mathematics is what we make it, . . . and having been made, it may at some future time even fail to be 'mathematics' any longer."[1] This simple, yet profound, observation by Professor Wilder justifies, if justification is indeed necessary, that important activity called the history of mathematics. And it also provides a clue to the use that we as teachers of mathematics can make of the history of mathematics, for the history of mathematics tells us what mathematicians made it in a given historical period and explains the influence of the presuppositions of that period.

The investigations of Hermann von Helmholtz (1821-1894) into the nature of geometrical axioms is a striking example of how the presuppositions of a historical period enter into the considerations of mathematical problems. Helmholtz typifies the radical empiricist of the nineteenth century, and his researches into the nature of geometric axioms furnish us with a picture of the empiricist's explication of geometrical axioms. As G. Stanley Hall, who was both a student and collaborator of Helmholtz before joining the faculty of Columbia University, has remarked, "The most radical side of Helmholtz's empiricism is to be found in his abstruse and largely mathematical discussions respecting the nature of the Euclidean axioms."[2] The following exposition will follow closely Helmholtz's lecture, "On the Origin and Significance of Geometrical Axioms."[3]

Helmholtz begins by asking what the origin of those propositions, which are unquestionably true yet incapable of proof, is in a science where everything else is a reasoned conclusion. He wonders if they were inherited from some divine source as the idealistic philosophers think. But yet another answer might be that the ingenuity of mathematicians has hitherto been unable to find proofs or at least explanations of these

[1] Raymond L. Wilder, *Introduction to the Foundations of Mathematics* (New York: John Wiley and Sons, 1952), p. 284.
[2] G. Stanley Hall, *Founders of Modern Psychology* (New York: D. Appleton and Co., 1912), p. 256.
[3] Hermann von Helmholtz, *Popular Lectures on Scientific Subjects* (London: Longmans, Green and Co., 1893), II, 27-75.

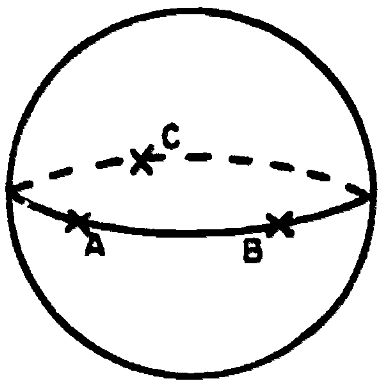


FIG. 1

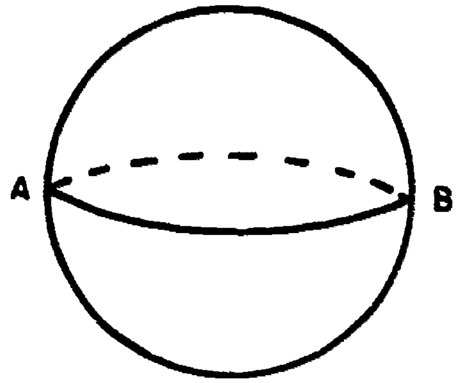


FIG. 2

propositions. As Helmholtz rightly asserts, "The main difficulty in these inquiries [the nature and origin of geometrical axioms] is, and always has been, the readiness with which results of everyday experience become mixed up as apparent necessities of thought with the logical processes, so long as Euclid's method of constructive intuition is exclusively followed in geometry."[4] The foundation of much proof in Euclid, as has been established by modern research, lies in the eighth axiom of Book I of *The Elements of Euclid* which states: "Magnitudes which coincide with another, that is which exactly fill the same space, are equal to one another."[5] The axiom, of course, expresses the intuitive notion of congruence. But as Helmholtz rightly maintains, if we build our necessities of thought upon this assumption, we must also assume the possibility of free translation of fixed figures with unchanged form to any part of space. And it is this latter assumption which we must examine since clearly no proof for it is given in Euclid. But let us make this latter point clear in some detail.

Imagine that we are beings of a two-dimensional space, that is, that we [illegible] and move on the surface of some solid body. Assume also that as rational beings we intend to develop a geometry for our two dimensional universe. As in Euclidean space, a point will describe a line and a line a surface; but if our universe were only two dimensional we would be incapable of imagining a surface moving out of itself just as we, in our three-dimensional space, are incapable of imagining a solid moving out of itself. But let us get back to our two-dimensional space. Given two points in that space, we could draw a shortest line between them; however, we must be careful to recognize that the shortest lines in our space

[4] *Ibid.*, p. 59.
[5] I. Todhunter (ed.), *The Elements of Euclid* (London: J. M. Dent and Sons Ltd., 1955), p. 6.

are not necessarily straight lines in the usual sense. The technical term for such lines is geodetic lines.

A geodetic line is characterized by the property that of all the paths on a given surface connecting any two points of the line, the shortest is the geodetic line itself. Clearly the straight line in ordinary Euclidean geometry is a geodetic line. We might visualize geodetics as essentially lines described by a tense (or taut) thread laid upon a surface so that it can slide freely upon that surface. Currently the term geodesic rather than geodetic is used for such lines as those described above, but in this discussion we shall use the term geodetic which follows exactly the usage of Helmholtz.*

To pinpoint this discussion further, let us suppose that we lived on the surface of the sphere illustrated in Figure 1. We ask what are our geodetics on this surface. As any navigator knows, they are the arcs of great circles. For example, the shortest distance from A to B is illustrated in Figure 1. It would be that arc from A to B lying along the great circle going through A and B. On the other hand, if we imagine a third point C, which is distinct from A and B, lying on the great circle through A and B and if we ask what is the shortest route from A to B (see Figure 1 for location of C), we see first of all that the shortest route no longer coincides with the geodetic since there are two geodetics, one going through C and the other not going through C. However, if A and B were to lie on the ends of a diameter, as in Figure 2, then there would be two geodetics and two shortest routes and they would not coincide. So we see that, at least on the surface of the sphere, it is not necessarily true that through every two points there is one and only one "shortest route."

Now suppose that we were dwellers on the surface of the sphere. We

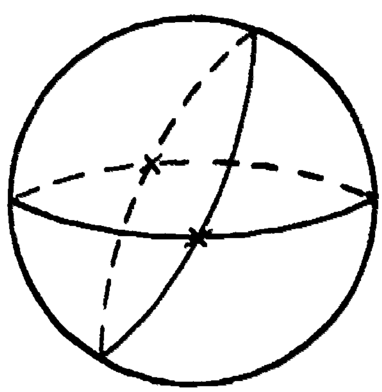


FIG. 3

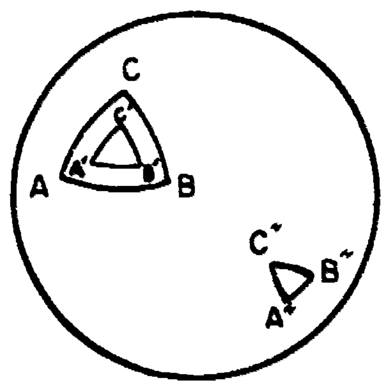


FIG. 4

* *Op. cit.*, p. 35.

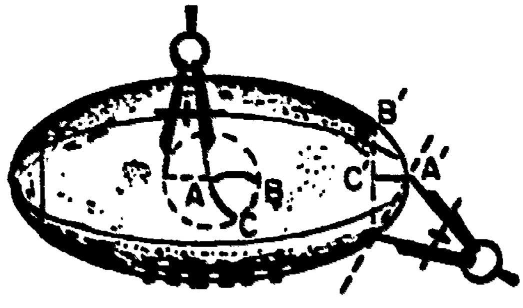


FIG. 5

would then know nothing of parallel lines (whereby parallel lines are meant nonintersecting geodetics), for clearly, as Figure 3 illustrates, any two distinct geodetics when continued will intersect in at least two points. Furthermore, we would have no notion of similarity, for as Figure 4 illustrates, the sum of the angles of $\Delta A'B'C'$ is less than the sum of the angles of ΔABC. We would, however, have a notion of congruence, i.e., again in Figure 4,

$$\Delta A''B''C'' \cong \Delta A'B'C'.$$

Thus we see that if we lived on the surface of a sphere, we would clearly set up a different system of geometric axioms from that which we are accustomed to, even though our logical powers might be exactly the same.

Now let us proceed to imagine ourselves as inhabitants of a different type of surface, say an ellipsoid, as in Figure 5. Here we could construct geodetics between any three points, but notice that triangles of equal geodetics, such as ΔABC and $\Delta A'B'C'$, no longer necessarily have equal angles. Clearly, there are distinct triangles of equal geodetic lengths which are congruent, but there are many which are not, depending on how near the pointed end or near the blunt end of our surface we have our triangles. Thus, on such a surface we could not move figures in any way we please, i.e., preserving geodetic distances, without changing their form. Some motions would preserve form, others would not. We might also add that circles of equal radii (length measured along geodetics) would not necessarily be the same, as is illustrated in Figure 5 by the circles traced out on the surface of the ellipsoid by the two compasses. The circumference would be greater at the blunter than at the sharper end.

Thus we see that if a surface admits of the figures lying on it being freely moved without change, the property is a special one which does not belong to every kind of surface. The condition under which a surface possesses this important property was first pointed out by Gauss. The measure of curvature, as Gauss called it, i.e., the reciprocal of the product of greatest and least curvature, must be everywhere equal over the whole extent of the surface.[7] Gauss also showed that this measure of curvature is not changed as the surface is bent without distention or contraction of any part of it. (By distention, Gauss meant stretching or distorting or tearing.) For example, a sheet could be wrapped into a cylinder and the properties dependent upon curvature would remain invariant, or stated otherwise, the geometry insofar as it depended on curvature would be the same on a cylindrical surface as it would be on a plane. Of course, there are certain niceties which must be taken care of in terms of the finiteness of the cylindrical surface but we shall pass by these difficulties at this time.

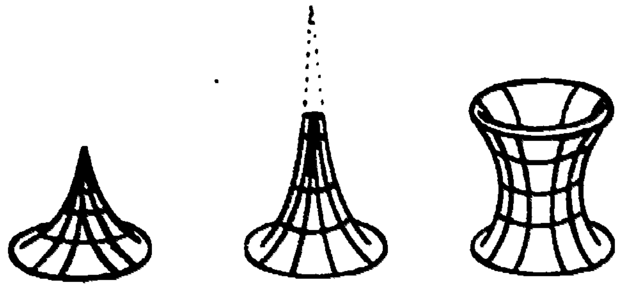


FIG. 6

We must next introduce a third type of surface, the surfaces which Helmholtz called "pseudospheres." These essentially can be any of the three illustrated in Figure 6. The important property of these pseudospheres, i.e., the surfaces of these pseudospheres, is that again the measure of curvature is constant, only it is negative instead of positive, so that the property of congruence holds on them. That is, all figures constructed at one place can be transferred to any other with perfect continuity of form and perfect equality of all dimensions lying on the surface itself. However, on these pseudospheres the parallel axiom no longer is valid. The difficulty here, however, is unlike that on the surface of the sphere, where no parallels existed. Here, given a geodetic and a point not on it, we can construct an infinite number of distinct parallels through the point to the given line, i.e., geodetics which will not meet the given geodetic regardless of how far they are extended. We, of course, now only call two of these nonintersecting geodetics parallels, but here

[7] The notion of curvature is essentially an analytic one which involves a discussion of the Gaussian spherical representation of a surface. The least and greatest curvatures are then related to the tangents and normals of the mapping. It is outside the limitations set for this paper to discuss this concept carefully. A discussion of it, however, can be found in Hilbert and Cohn-Vossen, *Geometry and the Imagination* (New York: Chelsea Publishing Co., 1952), pp. 193-97.

the important fact is the existence of more than one geodetic not intersecting the given geodetic.

Let us pause to sum up what we have discovered. On the surface of an ellipsoid or on a sphere, the parallel axiom does not hold since any two geodetics intersect; in an ordinary Euclidean plane there is exactly one line parallel to a given line through a given point; while on a pseudosphere given a line and a point not on the line there are infinitely many lines through the point that do not intersect the given line.

On the ellipsoid, the notion of congruence does not hold; on the sphere and on the pseudosphere the notion of congruence does hold; and in the Euclidean plane the notion of congruence holds.

On the ellipsoid, the notion of similarity does not hold. On the sphere and pseudosphere the notion of similarity does not hold. (In fact, both on the sphere and the pseudosphere the following theorem is true: Two triangles with three angles of one equal respectively to the three angles of the other are congruent. Hence there are no similar noncongruent triangles.) On the Euclidean plane the notion of similarity holds.

We might tabulate these results as follows:

Surface	*Parallelism*	*Congruence*	*Similarity*
Euclidean Plane	Yes (One through point to a line)	Yes	Yes
Surface of sphere	No	Yes	No
Surface of ellipsoid	No	No	No
Surface of pseudosphere	Yes (Infinitely many)	Yes	No

It is easy analytically to extend these four surfaces to three-dimensional spaces in which the properties which we have been discussing still do or do not hold, as the case may be, in their two-dimensional analogues; but, as Helmholtz himself admits, "When we pass to space of three dimensions, we are stopped in our power of presentation [visualization] by the structure of our organs and the experiences got through them which correspond only to the space in which we live."[8]

It is possible to represent the three-dimensional analogues of the above-discussed surfaces by means of the so-called Poincaré projective models, but the theory of projective models would take us far afield at the moment and certainly these models were unknown to Helmholtz. However, if the reader is interested, a thorough discussion of them can be found in the Hilbert Cohn-Vossen book, *Geometry and the Imagination*.[9]

[8] *Op. cit.*, p. 44.
[9] *Op. cit.*, pp. 242-63.

Helmholtz maintained, nevertheless, that it was possible to imagine conditions for which the properties or measurements of Euclidean space would become what they would be either in a spherical or pseudospherical space without the use of the above-referred-to-Poincare models. To understand his method, let us first remind ourselves that if the dimensions of other objects in our own world were diminished or increased in like proportion at the same time, we, with our means of space perception, would be utterly unaware of the change. This would also be the case if the distortion were different in different directions. The only proviso is that our own body be changed in the same manner simultaneously and that an object in rotating assumed at each instant the amount of distortion in its different dimensions corresponding to its position at that time.

For example, let us imagine the image of the world in a convex mirror. A well-made convex mirror represents objects to all as apparently solid objects and at fixed positions behind its surface. But, as we know, the images of the distant horizon and the sun in the sky lie behind the mirror at a finite distance which, in fact, is equal to the focal length of the mirror. Between the images of the distant horizon, say, and the surface of the mirror would be found the images of all other objects before it. But, of course, those images would be diminished and flattened in proportion to their respective distances from the mirror. The image of a man measuring with a ruler a straight line from the mirror would contract more and more the farther he went, but with his shrunken rule, the man in the image would count out exactly the same number of inches as the real man, and, in general, all geometric measurements whatsoever of lines and angles taken with regular varying images of real instruments would yield exactly the same results as in the outer world. Also, all congruent bodies would coincide on being applied to one another in the mirror as in the outer world. And all lines of sight in the outer world would be represented by straight lines of sight in the mirror.

But what does this prove? It shows that we cannot tell how the man in the mirror can discover that his body is not a rigid solid. It also shows that his experience (to him) is completely interpretable in terms of Euclid's axioms. Further, if he could look out upon our world as we can look into his, he would declare our world to be a picture in a spherical mirror and would speak of us just as we speak of him. And if he could communicate with us, Helmholtz concludes that neither he nor we could convince the other that he had the true and the other the distorted world picture.

Now if instead of a convex mirror we used a surface of a sphere, and if we imagined that in the sphere moving bodies contracted as they departed from the center as the images do in a convex mirror, we would have the "mirror analogue" of a pseudospherical space. And in a manner similar to that above, Helmholtz proceeds to analyze how objects in such a pseudospherical world, were it possible to enter one, would appear to an observer whose eye measure and experiences of space had been gained, like ours, in Euclidean space. In fact, he even shows the type of lens that we might use to have this visual experience. He says:

"Now we can obtain exactly similar images of our real world, if we look through a large convex lens of corresponding negative focal length, or even through a pair of convex spectacles if ground somewhat prismatically to resemble pieces of one continuous larger lens. With these, like the convex mirror, we see remote objects as if near to us, the most remote appearing no farther distant than the focus of the lens. In going about with this lens before the eyes, we find that the objects we approach dilate exactly in the manner I have described for pseudospherical space."[10]

Helmholtz finally concludes that he has shown how we can infer from the known laws of our own perceptions the sensations which a spherical or pseudospherical world would give if it existed. What necessarily follows is, of course, that we cannot hold that the axioms of geometry in any way depend upon a priori intuition. Helmholtz sums up his own argument as follows:

1. The axioms of geometry, taken by themselves out of all connection with mechanical propositions, represent no relations to real things. When thus isolated, if we regard them with Kant as forms of intuition transcendentally given, they constitute a form into which any empirical context whatever will fit, and which therefore does not in any way limit or determine beforehand the nature of the content. This is true, however, not only of Euclid's axioms, but also of the axioms of spherical and pseudospherical geometry.

2. As soon as certain principles of mechanics are conjoined with the axioms of geometry, we obtain a system of propositions which has real import, and which can be verified or overturned by empirical observations, just as it can be inferred from experience. If such a system were to be taken as a transcendental form of intuition and thought, there must be assumed a pre-established harmony between form and reality.[11]

The empirical foundation of our geometric axioms is thus established by answering positively the question: Can non-Euclidean geometry become an object of intuition? For Helmholtz if the foundations of geometrical axioms were not empirical, we should be unable to make non-

[10] *Op. cit.*, p. 61.
[11] *Ibid.*, p. 68.

Euclidean geometry the object of intuition. By (1) above, this intuition is limited to no particular geometric system, and by (2) this intuition, as well as its origins, assumes real import subject to empirical observation only when conjoined with certain principles of mechanics. Were this latter not the case, we would then have to assume that our intuition of space carried along with it an a priori knowledge of empirical data, i.e., objects and their spatial relations.

In conclusion, we might point out that the development of geometry has been a steady march toward abstractness — formalism in the modern terminology. This development has progressively forced empirical subject matter into the background, and in our own times has attempted to eliminate it from geometry altogether. To regard geometry, however, as a purely deductive science is to forsake entirely the view that geometry — considered as a branch of mathematics — depends in any way upon spatial intuition. And if we consider the current abstract point of view in geometry a commonplace notion, we should remind ourselves — as a partial critique of our own preconceptions — that the slow elimination of such empirical dependence in geometrical considerations was neither easy nor always correct, and the imaginative and resourceful work of Helmholtz shows us that it was no David who finally slew the empiricists. Unless, of course, David's last name was Hilbert.

FOREWORD

Strangely enough, fundamental or universal ideas are often the most difficult to define precisely. Have you ever tried to formulate a careful, logical definition of "straight line"? If you have, you will appreciate the difficulties.

Even dependence upon physical reality helps but little. As a noted scientist inadvertently once said:

> "No power on earth, however great,
> Can pull a string, however fine,
> Into a horizontal line
> That shall be absolutely straight."

Or, as another thoughtful observer has put it:

> "A straight line has no width, no depth, no wiggles, and no ends. There are no straight lines. We have ideas about these non-existent impossibilities: we even draw pictures of them. But they do not exist . . ."

In this essay the author explores some of the weaknesses of Euclid's basic definitions and axioms; the inherent difficulty, indeed, the impossibility, of defining rigorously all terms used in a given discipline; and the subtle implications of the concept of "straightness" for contemporary pure and applied mathematics.

The Straight Line

Euclid's definition of it is worthless, but his axioms remain basically sound. His successors in mathematics have extended his ideas to the curved line and a clearer concept of length

Morris Kline

We often mistake familiarity for understanding. but of course they are not the same thing. For example, every wife is familiar with her husband but certainly does not understand him, as every husband will certify. Among the subjects we suppose we understand, nothing would seem less complicated than the straight line. It is so familiar and so obvious that it hardly seems worth talking about. And yet the fact is that mathematicians have found the straight line a most complex and subtle study, and it has taken hundreds of years of analysis by many brilliant minds to arrive at a full understanding of it as a logical concept.

The straight line is easy enough to picture in physical terms — e.g., a string stretched taut between two points, the edge of a ruler, and so on. But these devices do not answer the question: What is the mathematical straight line? The mathematical line has no thickness, no color, no molecular structure. It is an abstraction — an idealization of the ruler's edge and the stretched string. What properties does the mathematical straight line possess?

Euclid attempted to define it in this manner: "A straight line is a curve which lies evenly with the points on itself." He defined a curve (line) as length without thickness, and a point as something having no "parts." But he failed to define either length or part, so that his definition rests upon physical conceptions and is therefore not acceptable as a mathematical definition, for mathematical logic must be independent of physical meanings. Furthermore, the phrase "lies evenly with the points on itself" is completely mysterious. We must conclude that Euclid's definition is worthless.

If this is so, how was Euclid able to proceed with the construction of a logical system of geometry? The answer is that, as mathematicians now realize, any logical system must start with undefined concepts, and it is the axioms of the system, and only these, that specify the properties of all the concepts used in the proofs. Without being at all aware of this, Euclid did the right thing: he ignored his worthless definitions of point, curve and straight line, so that in effect these concepts were undefined, and he proceeded to state the 10 axioms of his geometry.

His axioms state, among other things, that a straight line is determined by two points, that it may have i.:definite length, that any two right angles are equal, that only one line parallel to a given line can pass through a point outside that line in the same plane, that when equals (e.g., equal line segments) are added to equals the sums are always equal, and that the whole is greater than any of its parts. From these axioms Euclid deduced hundreds of theorems which tell us much more about the mathematical straight line.

As we examine these axioms and theorems, we nod our heads in agreement and in approval. Euclid does seem to have described the essence of the straight line. The straightness of the ruler's edge and the stretched string are apparently bound fast in his system of geometry; in particular, the shortest distance between any two points in the space described by his geometry is a straight line. (Incidentally, this fact is not axiomatic, as commonly supposed, but is a deep mathematical theorem.)

But now, as we explore further, we meet a disturbing fact. Consider a curved surface such as the one shown on the next page [Figure 1]. The shortest path between two points on this surface is not a straight line in the usual intuitive sense. And yet the surprising fact is that such paths and figures formed by them on this surface obey all the axioms of Euclidean geometry! For example, the axiom of parallel lines applies here: given a curve which represents the shortest distance betwen two points on the surface (this curve is called a geodesic), we can draw only one geodesic through a point not on this curve which will never meet the first geodesic however far the two curves are extended.

The point of this model is that Euclid and all the mathematicians who accepted Euclidean geometry until recent times believed erroneously that the Euclidean axioms and theorems applied only to the straight lines formed by rulers' edges and stretched strings. But now we find that the axioms and theorems also apply perfectly to figures on a curved surface. The situation is analogous to that of a client who asks an architect to design a house according to certain instructions and then finds when the house is built that it does not look at all like what he had envisioned, although the architect obeyed his instructions as far as they went. The trouble is that his instructions were not sufficiently restrictive. So it is with Euclid's axioms. They give so much leeway that they describe curved lines as well as straight ones.

In the early years of the 19th century mathematicians, though not yet concerned with the flaws we have just examined, attacked the Euclidean axioms on other grounds. They were troubled by Euclid's assumption that a pair of lines might never meet however far they were extended in space. They believed that they should not postulate so boldly what happens in regions inaccessible to experience. Hence they sought to replace

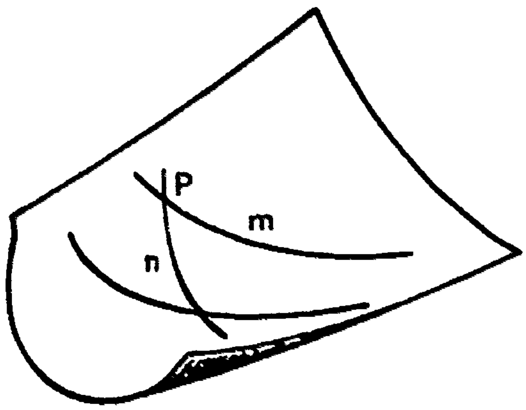


FIG. 1

this axiom by an axiom intuitively more acceptable or more readily verifiable. But every proposed substitute for the parallel axiom was found on analysis to involve assumptions about space as objectionable as the Euclidean parallel axiom. Whereupon some mathematicians adopted an entirely new course.

The first of these innovators, the Jesuit priest Girolamo Saccheri, had set out to prove that Euclid's axiom was the only possible correct one to showing that any other parallel axiom contradicting his would lead to a contradiction in the resulting system of geometry. First he was able to show that such contradictions would arise if one proposed the axiom that there is *no* parallel to a given line through a point not on this line. Then he examined the proposition that *more than one* parallel to the line might pass through such a point. In this case he arrived at no outright logical contradictions, but the theorems he derived were so strange that he concluded this system of geometry made no sense.

Saccheri, it turned out, had given in too easily. The system based on his second axiom was soon shown to be less absurd than it seemed. The great mathematicians Karl Friedrich Gauss, Nikolai Lobachevski and János Bolyai, working independently, created a non-Euclidean geometry, which is named for Lobachevski because he was the first to publish the results. This geometry was built, in effect, upon Saccheri's more-than-one-parallel axiom and the other nine Euclidean axioms.

In considering the Lobachevskian geometry the reader must remember that we are dealing with mathematical abstractions, not with physical objects. Some of the theorems and conclusions may seem at first sight to defy common sense, or the evidence of our senses. However, all we can ask of a logical system is that it be rigorously logical and consistent

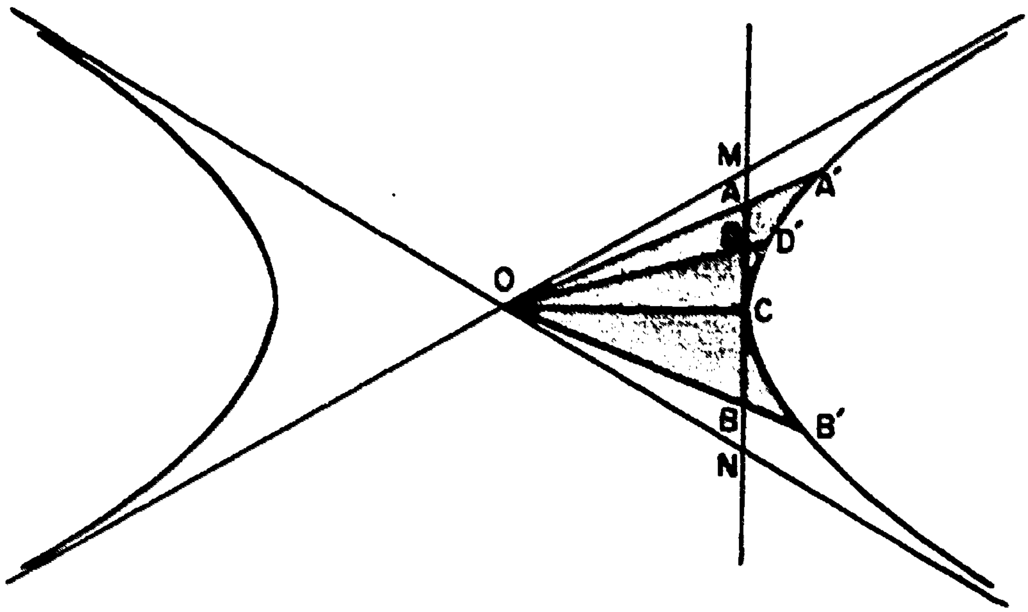


FIG. 2

within itself. As a matter of fact, it is found upon analysis that the new non-Euclidean geometries ag: ·:e with physical observations as well as Euclidean geometry does.

In this vein let us compare the Lobachevskian and Euclidean straight lines. We start with a line tangent to a hyperbola at the vertex [Figure 2]. In Lobachevskis' geometry, as a consequence of his axioms, the length of the segment AB of this line has a value which corresponds numerically to the area A'OB' in the Euclidean system. (The proof of this equality need not concern us here.) Similarly the line segment AC is equal to the area COA'. Now let us suppose that the area COD' equals the area D'OA'. This being so, in Lobachevskian geometry the line segment AD must be equal to the segment DC, although in Euclidean geometry their lengths are obviously different. There are still more remarkable consequences. As we move point *A* closer to *M*, the area COA' increases rapidly, and so does the numerical value of the line CA. If the line OM is tangent to the hyperbola at infinity (what is called an asymptote), then the area becomes infinite when *A* reaches *M*, and the length of the line CM likewise is infinite, according to the Lobachevskian geometry. Thus the entire straight line of Lobachevski's geometry, though infinite in length as far as his system is concerned, is represented within the finite Euclidean line MN.

What this comparison of the Euclidean and Lobachevskian lines teaches us is that the Euclidean mathematicians were parochial in their understanding of equality, or "congruence." The Euclidean axioms concerning the equality or inequality of line segments were framed with the concept of rigid bodies in mind. Euclid intended that equal segments be those which yielded the same lengths when measured by a rigid ruler, and mathematicians followed his lead. But the Lobachevskian line shows that line segments may be equal in spite of the fact that a rigid ruler indicates them to be unequal.

This fact suggests that even the Euclidean concept of equality may have entirely new physical interpretations. Suppose that as we moved out toward the ends of the universe our measuring rod contracted more and more, and all physical objects and distances shrank in the same proportion. We would be unaware of the contraction, and would believe that we were living in an infinitely extended world, although actually it might be finite, as measured by a truly rigid ruler. In other words, just as soon as we recognize that the Euclidean equality axioms can be satisfied with a new physical meaning for equality, we must recognize also that the physical world which appears to be Euclidean may actually possess quite a different structure.

Bernhard Riemann conceived another non-Euclidean geometry which

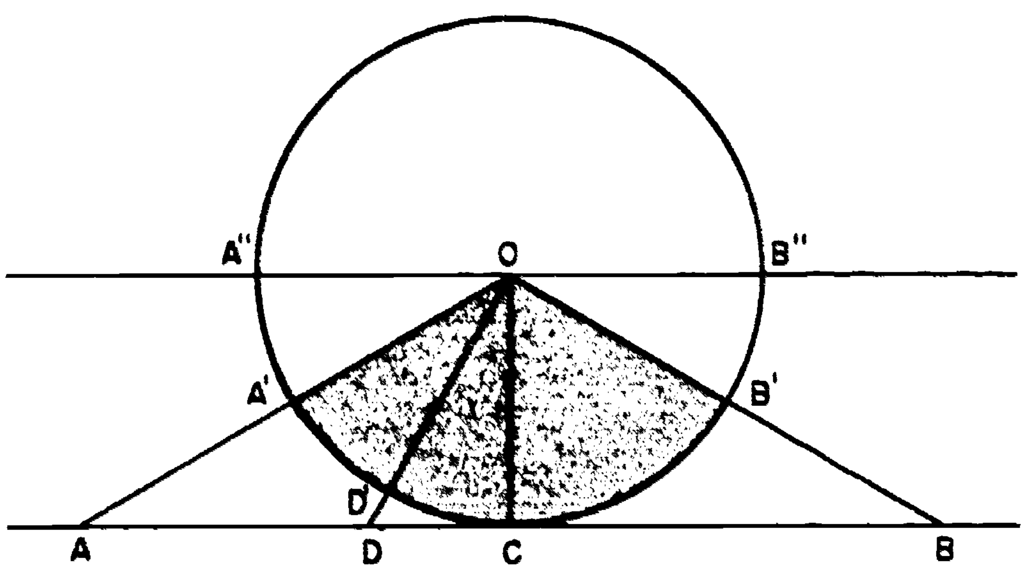


FIG. 3

took the opposite tack from Lobachevski's. He started with Saccheri's first proposition — that there are no parallel lines to a given line — and with the remaining nine axioms of Euclid, modified in minor respects. In Riemann's geometry the straight line proves to be *finite* in length and to have the structure of a circle.

We can illustrate the difference between the Riemannian line and the Euclidean line with another diagram [Figure 3]. Here a Euclidean line is tangent to a circle, and we consider the segment AB. In Riemann's geometry the length of AB turns out to be equal *numerically* to the Euclidean area A'OB' within the circle (the shaded area in the figure). As a consequence of this fact, if the area COD' equals the area D'OA', for example, the line segment CD is equal to DA, although by Euclidean standards the latter is much longer than the former. Now however far we may move *A* to the left and *B* to the right, the length of AB cannot exceed the area of the semicircle below A"B"; that is, the Riemannian line has finite length. The interpretation also suggests how an infinite world could be represented by a geometry with finite lines. Whereas with a contracting measuring rod a finite world might appear infinite, with an expanding measuring rod an infinite world would appear finite.

The farther we pursue the theorems of the non-Euclidean geometries, the more their straight lines affront our intuition and incite us to rebel. And yet, if we object to the finiteness of the Riemannian line and its circular structure on the ground that our view of the universe calls for lines extending indefinitely far out into space, a mathematician can reply that this notion is merely the product of an unbridled and untutored imagination. Within the actually observable world of our experience

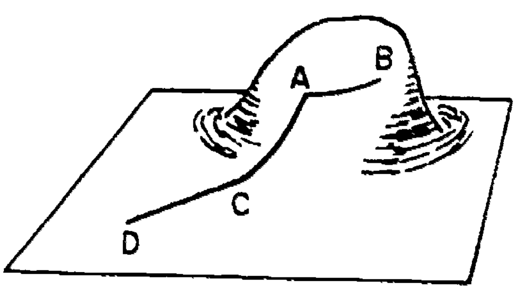


FIG. 4

the non-Euclidean geometries furnish an accurate description of physical realities.

The "straight line" in all geometries has the same basic definition: the shortest distance between two points. This is true whether we consider a stretched string, an arc on the surface of a sphere or even a more complex path. Let us take the case of a round hill standing on a flat plain [Figure 4]. The shortest path between two points around the brow of the hill (AB) may be the arc of a circle; the shortest path from *A* to *C* at the base of the hill may be a rather flattened S curve; from *C* to *D* on the plain we have an ordinary straight line. If one were to construct a geometry to fit this surface, the "straight" line of this geometry would have to have the properties common to AB, AC and CD—that is, the properties of the various geodesics on this surface. Obviously construction of such a geometry would not be simple.

In the geometry of the theory of relativity the paths of light rays in space-time are the geodesics, and these play the role of the straight line. The geodesics are generally not straight. In fact, this geometry possesses some of the peculiarities of a flat region containing hills. Each mass in

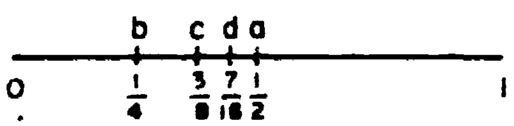


FIG. 5

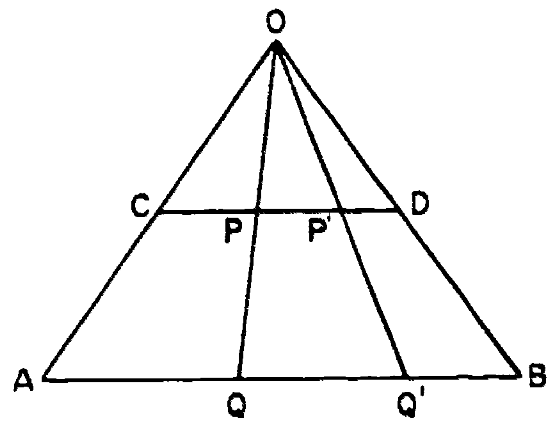


FIG. 6

space-time (e.g., the sun) acts like a hill causing the geodesics to depart from straightness.

Suppose now we turn from the vast realm of space to minute segments of the straight line. What does mathematics have to say about the internal structure of a segment?

Intuition and Euclidean geometry agree that we may divide this segment into any given number of equal parts. If we label the midpoint ½, then marked the quarters, divided the segment again into eighths [Figure 5] and continued unendingly to halve the successively smaller segments, it would appear that eventually we could label every point on the line. A moment's thought makes clear, however, that this is impossible. For one thing, there is an infinite number of fractional lengths not included in the foregoing set (e.g., 1/3, 1/5). For another, we must also consider fractional lengths corresponding to the irrational numbers (half the square root of 2, one-third the square root of 2, and so on) which are also infinite in number and terminate at other points on the line. As a matter of fact, the unit segment contains more irrational points than rational points. The entire collection of points on the segment is called the continuum — the "grand continuum" in the words of the mathematician J. J. Sylvester.

Now consider a line twice as long as the preceding unit. One would expect it to contain twice as many points. But paradoxically it has exactly

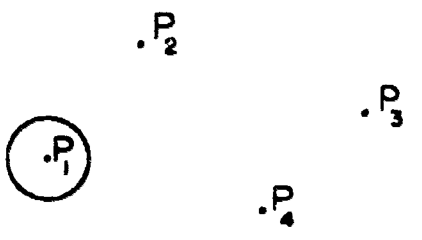


FIG. 7

the same number of points. This is easily proved by forming a triangle with the longer line as the base and the shorter one as the midline [Figure 6]. Now if we draw lines from the apex of the triangle through the midline to the baseline, we can see that for every point crossed in the former, there will be a unique corresponding point in the latter. Thus there is a one-to-one correspondence between the points on CD and on AB. But one-to-one correspondence is precisely the basis for asserting the equality of the number of objects in two sets of objects. If an army of soldiers each carrying a gun were to pass in review before us, we would know at once that there are as many guns as soldiers.

If all line segments contain the same number of points, how can they differ in length? This question was raised in a more general form more than 2,000 years ago by the Greek philosopher Zeno. Magnitude, said Zeno, must be divisible. Points, being indivisible, can possess no magnitude. A line segment, then, being made of points which have no magnitude, cannot itself have magnitude, any more than a noise can be a composite of silences. In other words, how can length arise from a conglomeration of points which have no length?

Modern mathematics has taken up and answered this question by introducing the "theory of measure," by which, through assigning lengths to rather arbitrary sets of points on a line, the paradox is resolved. We need not go into the process here; it is sufficient to say that the method has enabled us to penetrate somewhat into the murky darkness of the interior of the straight line.

The straight line is the starting point for another deep investigation which has come to fruition within the last quarter century. The subject in question was clearly in Euclid's mind when he defined a curve (line) as breadthless length and then stated that its extremities are points. He was saying in effect that the line is one-dimensional and the point zero-dimensional. The line is not a band or strip. But breadthless length, we saw, is a physical definition and hence not acceptable to mathematics. Just what do we mean by the intuitive or physical statement that a line is one-dimensional?

The full answer is a long story, but the essential idea is as follows. Let us ask the more general question: What shall we mean by the dimension of any set of points in a Euclidean plane? To begin with, suppose that we have a set of points whose dimension is to be determined. We surround the points of the set by small circles. If by making the circles sufficiently small we can avoid intersecting any points, then the set is said to be zero-dimensional. Thus, to take a trivial example, if the given set consists of a finite number of distinct points, we could certainly surround all these points by arbitrarily small circles which do not run through any point of the set [Figure 7]. Hence a finite set of points is zero-dimensional. Likewise the infinite set of points whose labels are 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, . . . is zero-dimensional [Figure 8].

We can now define one-dimensional sets in terms of zero-dimensional sets. A set of points, whether on a line or in a plane, is one-dimensional if it is not zero-dimensional and if arbitrarily small circles surrounding

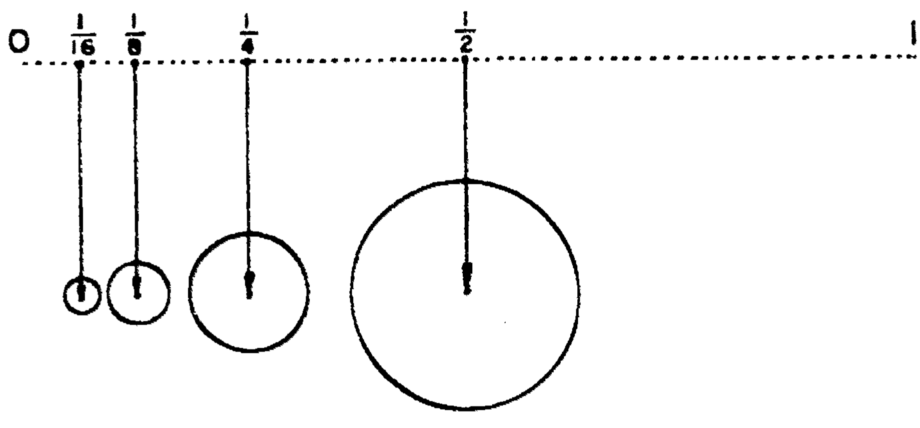


FIG. 8

each point of the set can be found which cut the set in a zero-dimensional set of points. Thus if we consider a straight line as the set of points whose dimension is to be determined, then we note that any circle surrounding a point of the line cuts the line in two points [Figure 9]. Since the two points of intersection are a zero-dimensional set by definition, and since any circle surrounding any point of the line must intersect the line, the line is one-dimensional.

To clarify the concept still further, let us apply it to the set of points in a square. If we surround any point inside the square with a small

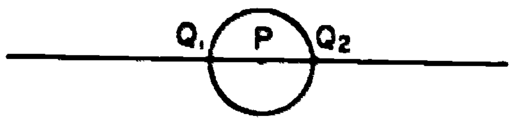


FIG. 9

circle, the entire circle will intersect the square [Figure 10]. If we surround any point on the boundary of the square with a small circle, then an arc of the circle will intersect the square [Figure 11]. Now a circle or any arc of the circle is one-dimensional by the definition of one-dimensional sets of points. Hence the points of the square are said to be a two-dimensional set.

Just to test the definition let us apply it once more to a complex curve — the outline of a four-petaled rose [Figure 12]. The most complicated portion of this curve centers on the middle point. If we surround this point with an arbitrarily small circle, the circle will cut the curve in eight points. Since this intersection is a finite set of points, the intersection is zero-dimensional and the curve itself, therefore, one-dimensional.

FIG. 10

FIG. 11

Some of the principal features of the internal organization of the straight line are now before us. It is to be hoped that they have given some understanding of that structure. This structure provides the an-

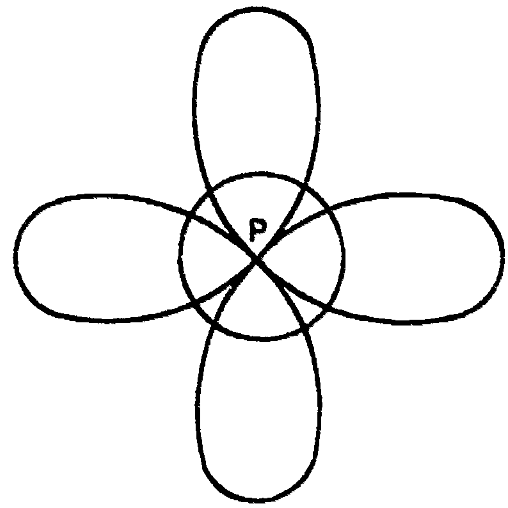


FIG. 12

swers to other problems involving the straight line. Let us listen once more to Zeno, the master of paradoxes. An arrow in flight, he says, is at any one moment in a definite position and therefore at rest. Hence the flying arrow is at rest wherever it is throughout its flight! With these words Zeno struck at the very concept of motion.

We must agree with Zeno that at any instant of its flight the moving arrow is somewhere in a definite position. And we may, if we like, even agree that the arrow is at rest in each of these positions. How then can the arrow pursue its smooth, continuous course? The structure of the straight line supplies the answer. It tells us that a continuous segment is composed of an infinite number of densely packed points. So continuous motion is no more than an assemblage of densely packed positions of rest. It is like a motion picture with an infinite number of still shots.

Our purpose in this brief survey has been to show that the seemingly simple problem of the nature and structure of the straight line leads to large modern developments in mathematics. These developments may leave readers with the impression that mathematics has gone to fantastic lengths and has distorted the original intuitive concept beyond recognition. Admittedly mathematics is a creation of the mind and its reality is not the reality of the physical world. Yet just these seemingly unrealistic idealizations of mathematics have proved to be the foundation and strength of modern science, as well as of mathematics. More than that, they are the bridge between the world of the senses and any reality that man may come to know.

FOREWORD

During the early part of the nineteenth century, one of the most prominent of European philosophers, Immanuel Kant, promulgated rather positive views concerning the nature of space. According to Kant, space was "a pure form of sensuous intuition." What he meant is explained in his own words: "Time and space are two sources of knowledge from which various *a priori* synthetical cognitions can be derived. Of this, mathematics gives a splendid example in the case of our cognition of space and its various relations. As they are both pure forms of sensuous intuition, they render synthetic propositions *a priori* possible."

Thus from Kant's point of view, mathematical "truths" or relations are neither invented nor created; since mathematics exists *a priori*, they are merely discovered (or rediscovered).

With the creation of non-Euclidean geometry about the middle of the nineteenth century, Kant's notions of the nature of mathematics and of space were thoroughly discredited. Henceforth the notion of "intuition" in mathematics took on a completely different aspect. The mathematician of today no longer regards mathematics as something "pre-existing." To be sure, he often uses physical models to *suggest* abstract concepts and relations, and it is chiefly in this sense that we think of the "intuitional basis" of modern mathematics.

Geometry and Intuition

A classic description of how "common sense," once accepted as the basis of physics but now rejected, is also inadequate as a foundation for mathematics

We have grown so accustomed to the revolutionary nature of modern science that any theory which affronts common sense is apt to be regarded today as half proved by that very fact. In the language of science and philosophy the word for common sense is intuition — it relates to that which is directly sensed or apprehended. Twentieth-century discoveries have dealt harshly with our intuitive beliefs about the physical world. The one area that is commonly supposed to remain a stronghold of intuition is mathematics. The Pythagorean theorem is still in pretty good shape; the self-evident truths of mathematics are in the main still true. Yet the fact is that even in mathematics intuition has been taking a beating. Cornered by paradoxes — logical contradictions — arising from old intuitive concepts, modern mathematicians have been forced to reform their thinking and to step out on the uncertain footing of radically new premises.

Some years ago the brilliant Austrian mathematician Hans Hahn surveyed the situation in a Vienna Circle lecture which he titled "The Crisis in Intuition." His analysis is still fresh and timely, and it is published here, in part, for the first time in English. Hahn began with Immanuel Kant, the foremost exponent of the importance of intuition, and showed how the foundations of Kant's ideas about knowledge "have been shaken" by modern science. The intuitive conceptions of space and time were jolted by Einstein's theory of relativity and by advances in physics which proved that the location of an event in space and time cannot be determined with unlimited precision. Hahn went on to consider the demolition of Kant's ideas about mathematics, and he illustrated his theme with the case of geometry, where "intuition was gradually brought into disrepute and finally was completely banished." This section of his lecture, somewhat condensed, follows.

Hans Hahn

One of the outstanding events in [the banishment of intuition from geometry] was the discovery that, in apparent contradiction to what had previously been accepted as intuitively certain, there are curves that possess no tangent at any point, or — what amounts to the same thing — that it is possible to imagine a point moving in such a manner that at no instant does it have a definite velocity. The questions involved here directly affect the foundations of the differential calculus as developed by Newton and Leibnitz.

Newton calculated the velocity of a moving point at the instant t as the limiting value approached by the average velocity between t and an instant close to it, t', as t' approaches t without limit. Leibnitz similarly declared that the slope of a curve at a point p is the limiting value approached by the average slope between p and a nearby point p' as p' approaches p without limit.

Now one asks: Is this true for every curve? It is indeed for all the old

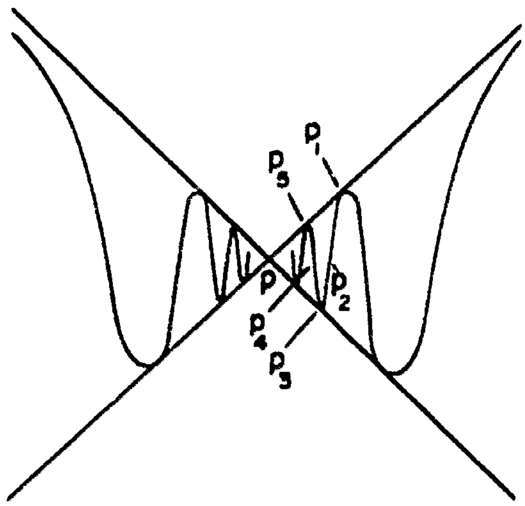


FIG. 1

familiar ones—circles, ellipses, hyperbolas, parabolas, cycloids, etc. But it is not true, for example, of a wave curve such as is shown here [*Figure 1*]. In the neighborhood of the point p the curve has infinitely many waves. The wavelength and the amplitude of the waves decrease without limit as they approach p. If we take successive points closer and closer to p, the average slope between p and p' (the moving point) drops from plus 1 through 0 to minus 1 and then rises from minus 1 to plus 1. That is, as p' approaches p without limit through infinitely many waves, the average slope between p and p' keeps oscillating between the values 1 and -1. Thus there can be no question of a limit or of a definite slope of the curve at the point p. In other words, the curve we are considering has no tangent at p.

This relatively simple illustration demonstrates that a curve does not have to have a tangent at every point. Nevertheless it used to be thought, intuitively, that such a deficiency could occur only at exceptional points of a curve. It was therefore a great surprise when the great Berlin mathematician Karl Weierstrass announced in 1861 a curve that lacked a precise slope or tangent at *any* point. Weierstrass invented the curve by an intricate and arduous calculation, which I shall not attempt to reproduce. But his result can today be achieved in a much simpler way, and this I shall attempt to explain, at least in outline.

We start with a simple figure which consists of an ascending and a descending line [*Figure 2*]. We then replace the ascending line with a broken line in six parts, first rising to half the height of the original

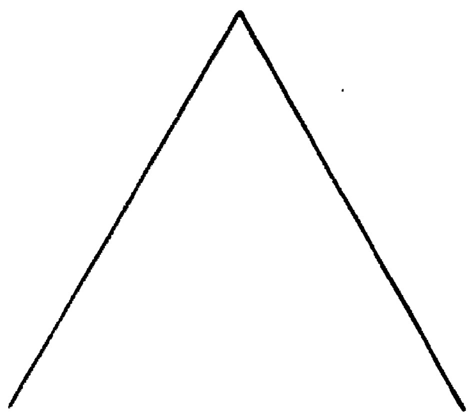


FIG. 2

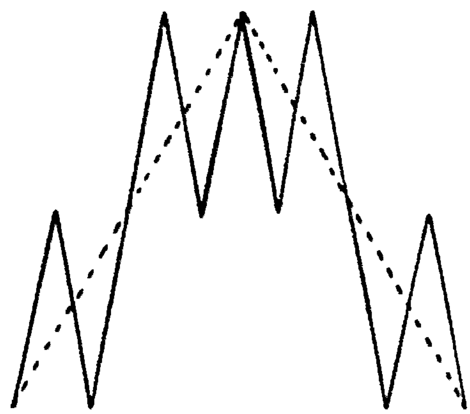


FIG. 3

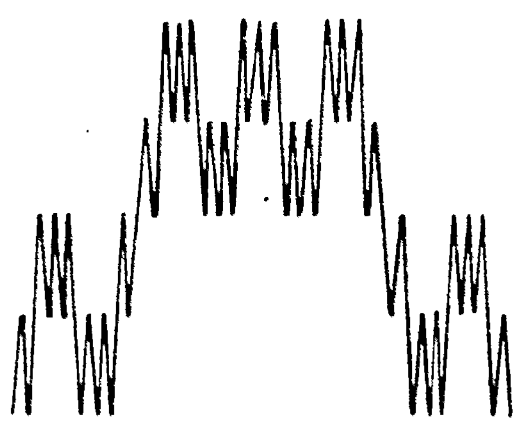


FIG. 4

line, then dropping all the way down, then again rising to half the height, continuing to full height, dropping back again to half height and finally rising once more to full height [*Figure 3*]. We replace the descending line also with a broken line of six similar parts. From this figure of 12 line segments we evolve, again by replacing each segment with a broken line of six parts, a figure of 72 line segments [*Figure 4*]. It is easy to see that repetition of this procedure will lead to more and more complicated figures. It can be demonstrated that the geometric objects constructed according to this rule approach without limit a definite curve possessing the desired property; namely, at no point will it have a precise slope, and hence at no point a tangent. The character of this curve of course entirely eludes intuition; indeed, after a few repetitions of the segmenting process the evolving figure has grown so intricate that intuition can scarcely follow. The fact is that only logical analysis can pursue this strange object to its final form.

Lest it be supposed that intuition fails only in the more complex branches of mathematics, I propose now to examine a failure in the elementary branches. At the very threshold of geometry lies the concept of the curve; everyone believes that he has an intuitively clear notion of what a curve is. Since ancient times it has been held that this idea

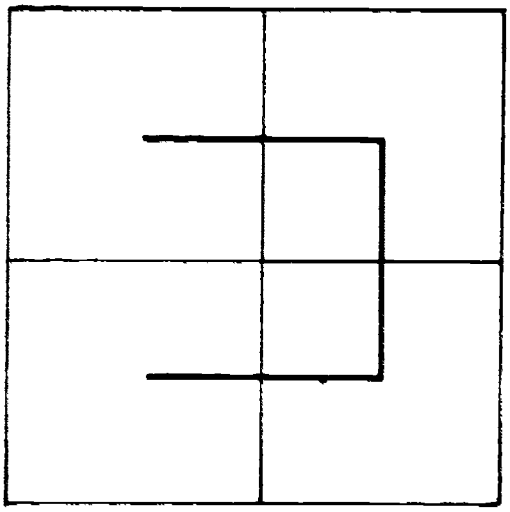


FIG. 5

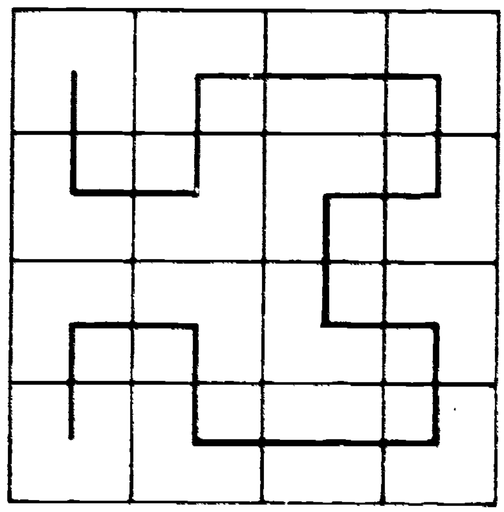


FIG. 6

could be expressed by the following definition: Curves are geometric figures generated by the motion of a point. But attend! In the year 1890 the Italian mathematician Giuseppe Peano (who is also renowned for his investigations in logic) proved that the geometric figures that can be generated by a moving point also include *entire plane surfaces*. For instance, it is possible to imagine a point moving in such a way that in a finite time it will pass through all the points of a square – and yet no one would consider the entire area of a square as simply a curve. With the aid of a few diagrams I shall attempt to give at least a general idea of how this space-filling motion is generated.

Divide a square into four small squares of equal size and join the center points of these squares by a continuous curve composed of straight-line segments [*Figure 5*]. Now imagine a point moving at uniform velocity so that it will traverse the continuous curve made of these line segments in a certain unit of time. Next divide each of the four squares again into four equal squares so that there are 16 squares, and connect their center points [*Figure 6*]. Imagine the point moving so that in the same time as before it will traverse this second curve at uniform velocity. Repeat the procedure, each time imagining the point to move so that in the same unit of time it will traverse the new system of lines at a uniform velocity. Figure 7 shows one of the later stages, when the original square has been divided into 4,096 small squares. It is now possible to give a rigorous proof that the successive motions considered here approach without limit a curve that takes the moving point through all the points of the large square in the given time. This motion cannot possibly be grasped by intuition; it can only be understood by logical analysis.

While a geometric object such as a square, which no one regards as a curve, can be generated by the motion of a point, other objects which one would not hesitate to classify as curves cannot be so generated. Observe, for instance, the wave curve shown here [*Figure 8*]. In the neighborhood of the line segment *ab* the curve consists of infinitely many waves whose lengths decrease without limit but whose amplitudes do not decrease. It is not difficult to prove that this figure, in spite of its linear character, cannot be generated by the motion of a point, for no

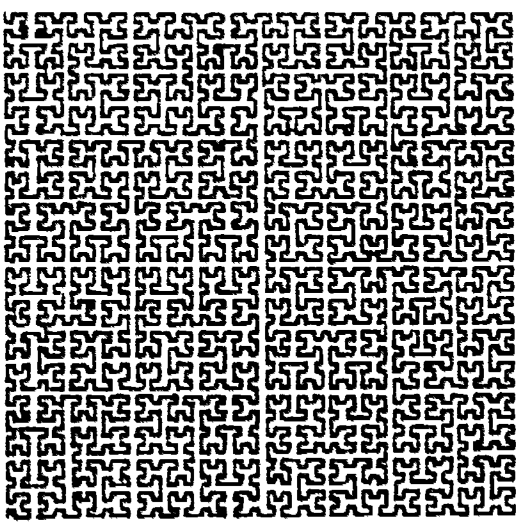


FIG. 7

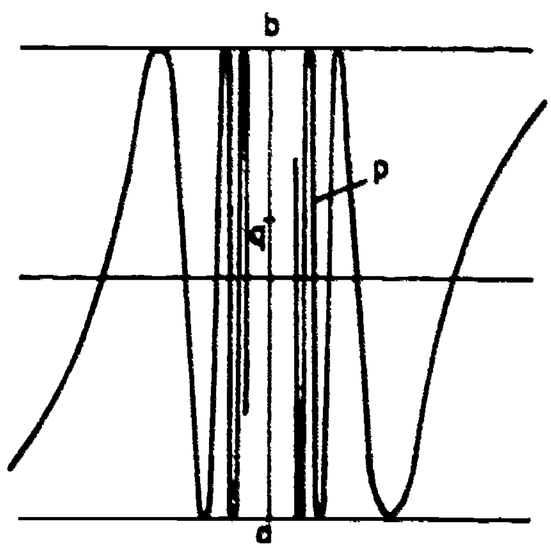


FIG. 8

motion of a point is conceivable that would carry it through all the points of this wave curve in a finite time.

Two important questions now suggest themselves. (1) Since the time-honored definition of a curve fails to cover the fundamental concept, what serviceable definition can be substituted for it? (2) Since the class of geometric objects that can be produced by the motion of a point does not coincide with the class of all curves, how shall the former class be defined? Today both questions are satisfactorily answered; I shall defer for a moment the answer to the first question and speak briefly about the second. This was solved with the aid of a new geometric concept—"connectivity in the small." Consider a line, a circle or a square. In each of these cases, we can move from one point on the figure to another very close to it along a path which does not leave the confines of the figure, and we remain always in close proximity to both points. This is the property called "connectivity in the small." Now the wave curve we have just considered does not have this property. Take for example the neighboring points p and q [*Figure 9*]. In order to move from p to q without leaving the curve it is necessary to traverse the infinitely many waves lying between them. The points on this path are not all in close proximity to p and q, for the waves all have the same amplitude.

It is important to realize that "connectivity in the small" is the basic characteristic of figures that can be generated by the motion of a point.

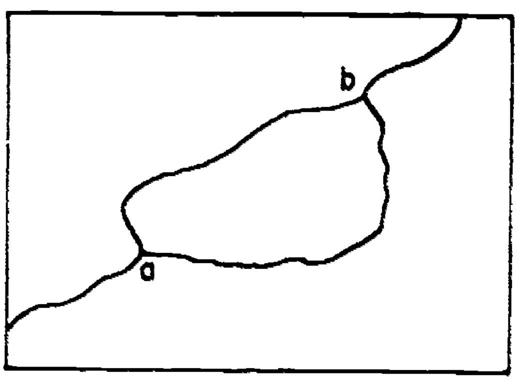


FIG. 9

A line, a circle and a square can be generated by the motion of a point because they are connected in the small; the wave figure shown cannot be generated by the motion of a point because it is not connected in the small.

We can convince ourselves of the undependability of intuition, even as regards such elementary geometrical questions, with a second example. Think of a map of three adjoining countries [*Figure 9*]. There are certain points at which all three countries come together—so-called "three-country corners" (points *a* and *b*). Intuition seems to indicate that such corners can occur only at isolated points, and that at the great majority of boundary points on the map only two countries will be in contact. Yet the Dutch mathematician L. E. J. Brouwer showed in 1910 how a map can be divided into three countries in such a way that all three countries will touch one another at every boundary point!

Start with a map of three countries—one hatched (A), one dotted (B)

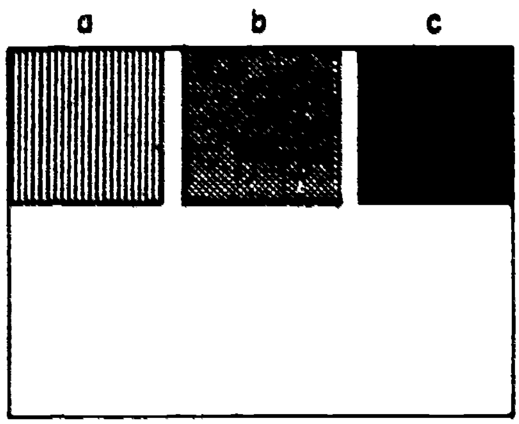


FIG. 10

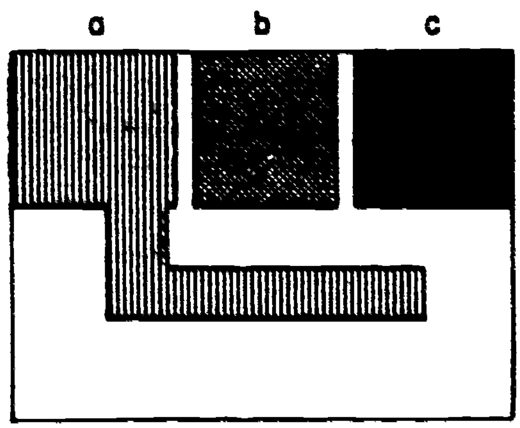


FIG. 11

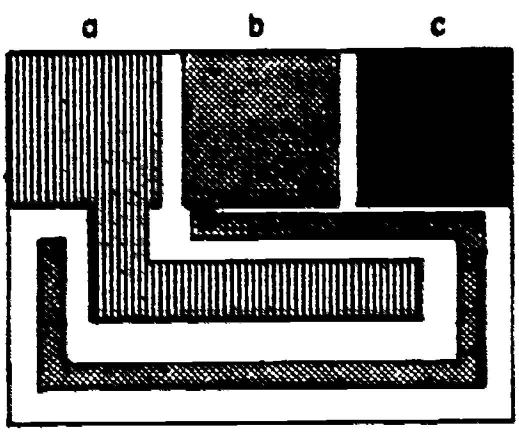


FIG. 12

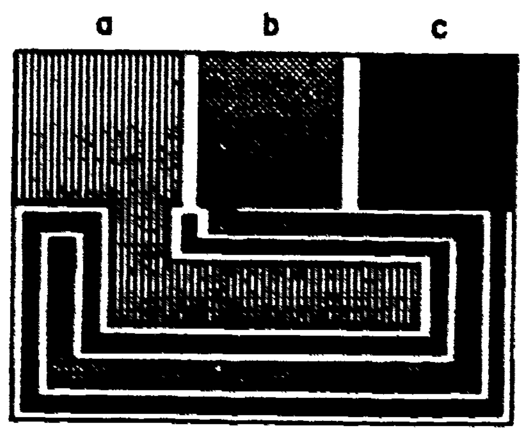


FIG. 13

and one solid (C)—and an adjoining unoccupied area [*Figure 10*]. Country A, seeking to bring this land into its sphere of influence, decides to push out a corridor which approaches within one mile of every point of the unoccupied territory but—to avoid trouble—does not impinge upon either of the two other countries [*Figure 11*]. After this has been done, country B decides that it must do the same and proceeds to drive into the remaining unoccupied territory a corridor that comes within one-half mile of all the unoccupied points but does not touch either of the other two countries [*Figure 12*]. Thereupon country C decides that it cannot lag behind, and it also extends a corridor into the territory as yet unoccupied, which comes to within a third of a mile of every point of this territory but does not touch the other countries [*Figure 13*]. Country A now proceeds to push a second corridor into the remaining unoccupied territory, which comes within a quarter of a mile of all points of this teritory but does not touch the other two countries. The process continues: Country B extends a corridor that comes within a fifth of a mile of every unoccupied point; country C, one that comes within a sixth of a mile of every unoccupied point; country A starts over again, and so on and on. And since we are giving imagination free rein, let us assume further that country A required a year for the construction of its first corridor, country B, the following half-year for its first corridor, country C, the next quarter year for its first corridor; country A, the next eighth of a year for its second, and so on, each succeeding extension being completed in half the time of its predecessor. It can be seen that after two years none of the originally unoccupied territory will remain unclaimed; moreover the entire area will then be divided among the three countries in such a fashion that all three countries will meet at every boundary point. Intuition cannot comprehend this pattern, but logical analysis requires us to accept it.

Because intuition turned out to be deceptive in so many instances, and because propositions that had been accounted true by intuition were repeatedly proved false by logic, mathematicians became more and more skeptical of the validity of intuition. The conviction grew that it was unsafe to accept any mathematical proposition, much less to base any mathematical discipline on intuitive convictions. Thus a demand arose for the expulsion of intuition from mathematical reasoning and for the complete formalization of mathematics. That is to say, every new mathematical concept was to be introduced through a purely logical definition; every mathematical proof was to be carried through by strictly logical means. The pioneers of this program (to mention only the most famous) were Augustin Cauchy (1789–1857), Bernhard Bolzano (1781–1848), Karl Weierstrass (1815–1897), Georg Cantor (1845–1918) and Julius Wilhelm Richard Dedekind (1831–1916).

The task of completely formalizing or logicizing mathematics was arduous and difficult; it meant nothing less than a root-and-branch reform. Propositions that had been accepted as intuitively evident had to be painstakingly proved. As the prototype of an *a priori* synthetic judgment based on pure intuition Kant expressly cited the proposition that space is three-dimensional. But by present-day standards even this statement calls for searching logical analysis. First it is necessary to define purely logically what is meant by the "dimensionality" of a geometric figure, and then it must be proved logically that the space of ordinary geometry—which is also the space of Newtonian physics—as embraced in this definition is in fact three-dimensional. This proof was not achieved until 1922, and then simultaneously by the Vienna mathematician K. Menger and the Russian mathematician Pavel Uryson (who later

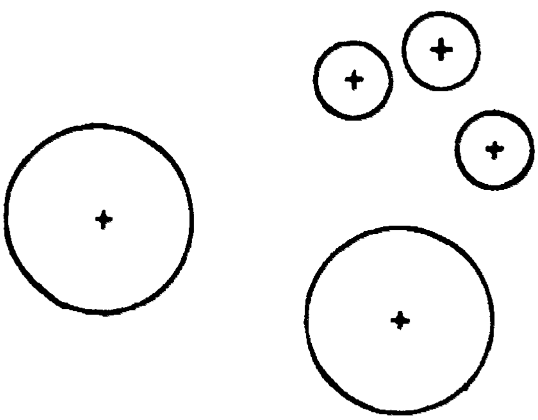


FIG. 14

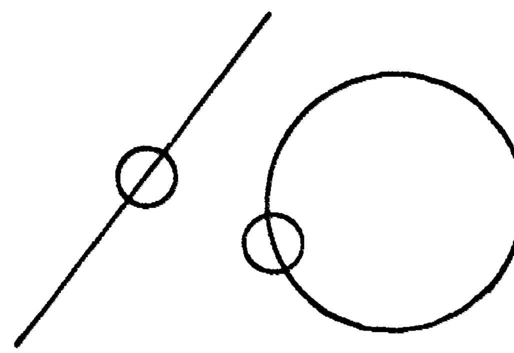


FIG. 15

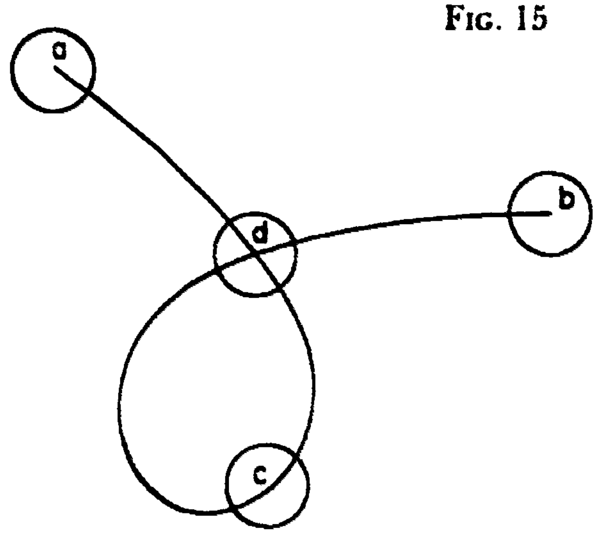


FIG. 16

succumbed to a tragic accident at the height of his creative powers). I wish to explain briefly how the dimensionality of a figure is defined.

A geometric figure is called a "point set." It is said to be null-dimensional if for each of its points there exists an arbitrarily small neighborhood whose boundary contains no point of the set. For example, every set consisting of a finite number of points is null-dimensional, but there are also many complicated null-dimensional points which consist of infinitely many points [*Figure 14*]. A point set that is not null-dimensional is called one-dimensional if for each of its points there is an arbitrarily small neighborhood whose boundary has only a null-dimensional set in common with the point set [*Figure 15*]. Every straight line, every figure composed of a finite number of straight lines, every circle, every ellipse — in short, all geometrical constructs that we ordinarily designate as curves — are one-dimensional in this sense. A point set that is neither null-dimensional nor one-dimensional is called two-dimensional if for each of its points there is an arbitrarily small neighborhood whose boundary has at the most a one-dimensional set in common with the point set. Every plane, every polygonic or circular area, every spherical surface — in short, every geometric construct ordinarily classified as a surface — is two-dimensional in this sense. A point set that is neither null-dimensional, one-dimensional nor two-dimensional is called three-dimensional if for each of its points there is an arbitrarily small neighborhood whose boundary has at most a two dimensional set in common with the point set. It can be proved — not at all simply, however — that the space of ordinary geometry is a three-dimensional point set.

This theory provides what we have been seeking — a fully satisfactory definition of the concept of a curve. The essential characteristic of a curve turns out to be its one-dimensionality. But beyond that the theory also makes possible an unusually precise and subtle analysis of the structure of curves, about which I should like to comment briefly.

A point on a curve is called an end point if there are arbitrarily small neighborhoods surrounding it, each of whose boundaries has only a single point in common with the curve [*points* a *and* b *in Figure 16*]. A point on the curve that is not an end point is called an ordinary point if it has arbitrarily small neighborhoods each of whose boundaries has exactly two points in common with the curve [*point* c *in Figure 16*]. A point on a curve is called a branch point if the boundary of any of its arbitrarily small neighborhoods has more than two points in common with the curve [*point* d *in Figure 16*]. Intuition seems to indicate that it is impossible for a curve to be made up of nothing but end points or branch points. As far as end points are concerned, this intuitive conviction has been confirmed by logical analysis, but as regards branch points it has been refuted. The Polish mathematician W. Sierpinski proved in 1915 that there are curves *all of whose points are branch points*. Let us attempt to visualize how this comes about.

Suppose that an equilateral triangle has been inscribed within another equilateral triangle and the interior of the inscribed triangle erased [*Figure 17*]. In each of the three remaining triangles [*the unhatched* *areas*] inscribe an equilateral triangle and again erase its interior; there are now nine equilateral triangles together with their sides [*Figure 18*]. Imagine this process continued indefinitely. (Figure 19 shows the fifth step, where 243 triangles remain.) The points of the original equilateral

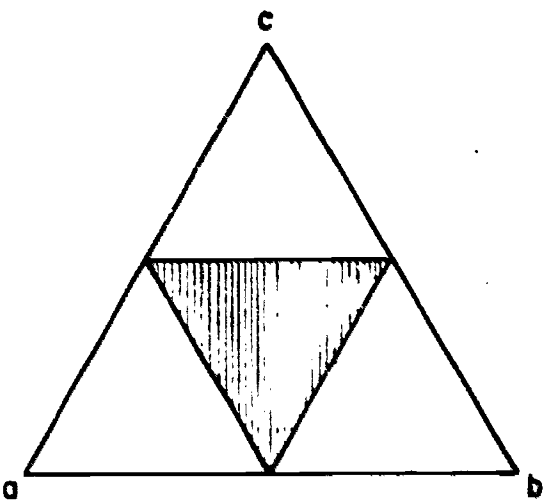


FIG. 17

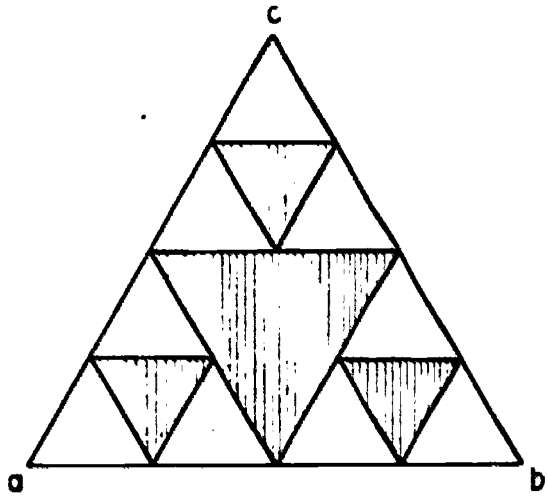


FIG. 18

triangle that survive the infinitely numerous erasures can be shown to form a curve all of whose points, with the exception of the vertex points *a*, *b* and *c* of the original triangle, are branch points. From this it is easy to obtain a curve with all its points branch points; for instance, by distorting the entire figure so that the three vertices of the original triangle are brought together in a single point.

But enough of examples—let us now summarize what has been said. Repeatedly we have found that in geometric questions, even in very simple and elementary ones, intuition is a wholly unreliable guide. And it is of course impossible to adopt this discredited aid as the basis of a mathematical discipline. The way is then open for other logical constructs in the form of spaces differing from the space of ordinary geometry; spaces, for instance, in which the so-called Euclidean parallel postulate is replaced by a contrary postulate (non-Euclidean spaces), spaces whose dimensionality is greater than three, non-Archimedean spaces (in which there are lengths that are greater than any multiple of a given length).

What, then, are we to say to the often-heard objection that the multidimensional, non-Euclidean, non-Archimedean geometries, though consistent as logical constructs, are useless in arranging our experience because they do not satisfy intuition? My first comment is that ordinary geometry itself is by no means a supreme example of the intuitive process. The fact is that *every* geometry—three-dimensional as well as multidimensional, Euclidean as well as non-Euclidean, Archimedean as well as non-Archimedean—is a logical construct. For several centuries, almost up to the present day, ordinary geometry admirably served the purpose of ordering our experience; thus we grew used to operating with it. This explains why we regard it as intuitive, and every departure from it contrary to intuition—intuitively impossible. But as we have seen, such "intuitional impossibilities" occur even in ordinary geometry. They appear as soon as we reflect upon objects that w h ot thought about before.

Modern physics now makes it appear appropriate to avail ourselves of the logical constructs of multidimensional and non-Euclidean geometries for the ordering of our experience. (Although we have as yet no indication that the inclusion of non-Archimedean geometry might prove

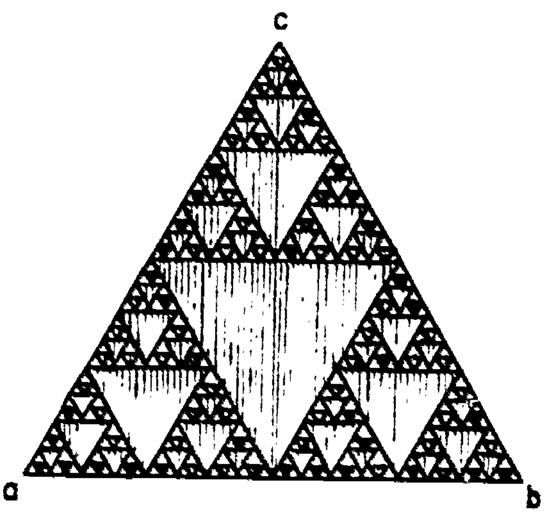


FIG. 19

useful, this possibility is by no means excluded.) But, because these advances in physics are very recent, we are not yet accustomed to the manipulation of these logical constructs; hence they are still considered an affront to intuition.

The same reaction occurred when the theory that the earth is a sphere was advanced. The hypothesis was widely rejected on the grounds that the existence of the antipodes was contrary to intuition; however, we have got used to the conception and today it no longer occurs to anyone to pronounce it impossible because it conflicts with intuition.

Physical concepts are also logical constructs, and here too we can see clearly how concepts whose application is familiar to us acquire an intuitive status which is denied to those whose application is unfamiliar. The concept "weight" is so much a part of common experience that almost everyone regards it as intuitive. The concept "moment of inertia," however, does not enter into most people's activities and is therefore not regarded by them as intuitive; yet among many experimental physicists and engineers, who constantly work with it, moment of inertia possesses an intuitive status equal to that generally accorded the concept of weight. Similarly the concept "potential difference" is intuitive for the electrical technician, but not for most people.

If the use of multidimensional and non-Euclidean geometries for the ordering of our experience continues to prove itself so that we become more and more accustomed to dealing with these logical constructs; if they penetrate into the curriculum of the schools; if we, so to speak, learn them at our mother's knee as we now learn three-dimensional Euclidean geometry — then it will no longer occur to anyone to say that these geometries are contrary to intuition. They will be considered as deserving of intuitive status as three-dimensional Euclidean geometry is today. For it is not true, as Kant urged, that intuition is a pure *a priori* means of knowledge. Rather it is force of habit rooted in psychological inertia.

FOREWORD

It is not easy to express in simple language the enormously significant contributions of Riemann to both pure mathematics as well as to mathematical physics. To say that Riemann, in the middle of the nineteenth century, literally revolutionized geometrical thought is scarcely an exaggeration.

Manifesting an intellectual courage of the highest order, Riemann convinced the mathematical world that no particular geometry nor any particular space was to be regarded as the necessary form of human perception. From that moment on the concept of "absolute space" was abandoned and the era of relativity was upon us.

The Curvature of Space

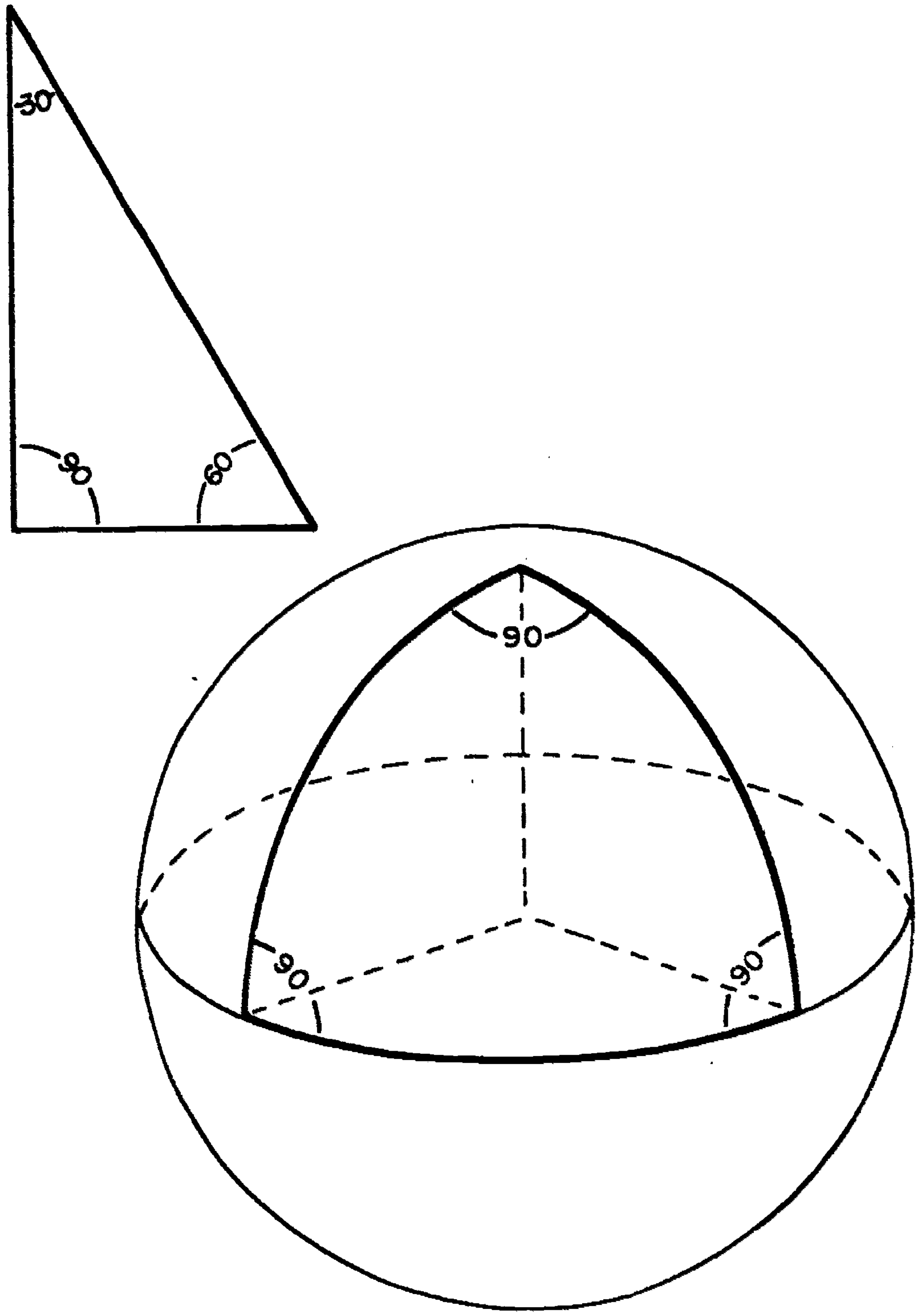
Just 100 years ago the young Bernhard Riemann gave his famed paper on the foundations of geometry. He discussed space of four or more dimensions, and paved the way for the General Theory of Relativity

P. LeCorbeiller

In the spring of 1854 a young German mathematician named Bernhard Riemann was greatly worried about his future and about a test he faced immediately. He was already 28, and still not earning—he was living meagerly on a few thalers sent each month by his father, a Protestant minister in a small Hanover town. He wrote modestly to his father and brother that the most famous university professors, in Berlin and in Göttingen, had unaccountably been extraordinarily kind to him. He had his doctor's degree; now, to obtain an appointment as a lecturer (without stipend), he had to give a satisfactory lecture before the whole Faculty of Philosophy at Göttingen. He had offered three subjects. "The two first ones I had well prepared," Bernhard wrote his brother, "but Gauss chose the third one, and now I'm in trouble."

Karl Friedrich Gauss was the dean of German mathematicians and the glory of his university. In Bernhard's picture of Heaven, Gauss's professorial armchair was not very far from the Lord's throne. (This is still the general view in Göttingen today.) The subject Gauss had chosen for young Riemann's lecture was "The Hypotheses That Are the Foundations of Geometry." Gauss had published nothing but a few cryptic remarks on this topic, but he selected it in preference to the two others proposed by Riemann because he was curious to find out what the young man would have to say on such a deep and novel subject—a subject to which Gauss himself had given much thought and had already made a great, though as yet not widely appreciated, contribution.

The day of Riemann's public lecture was Saturday, June 10, 1854. Most of his auditors were classicists, historians, philosophers—anyway, not mathematicians. Riemann had decided that he would discourse about the curvature of n-dimensional spaces without writing any equations.



TRIANGLES drawn on a plane and on a sphere obey somewhat different rules. On a plane the sum of the angles of a triangle always is equal to 180 degrees. The intersection of three great circles on the surface of a sphere forms three angles adding up to 270 degrees.

Was that a courteous gesture on his part, or a mildly Machiavellian scheme? We shall never know. What is sure is that without equations Gauss understood him very well, for walking home after the lecture he told his colleague Wilhelm Weber, with unwonted warmth, of his utmost admiration for the ideas presented by Riemann.

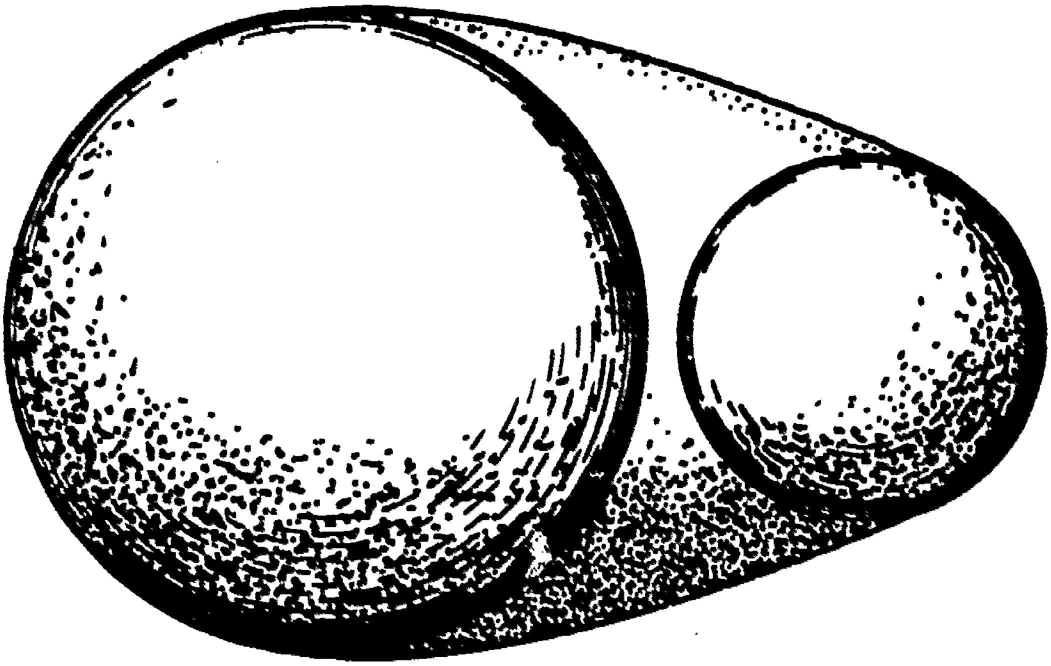
Gauss's enthusiasm was justified. The young man had reached into realms of thought so new that few scientists then could follow him. But his abstract ideas were to make contact with experimental reality half a century later through the work of Albert Einstein, who saw that Riemann's speculations were directly applicable to the problem of the interaction between light and gravitation, and made them the basis of his Generalized Relativity Theory, which today controls our view of the universe.

Let us then go back 100 years and acquaint ourselves with the thoughts which Riemann made public on that June day of 1854. Before reaching Riemann's ideas we first have to cover some rather elementary background.

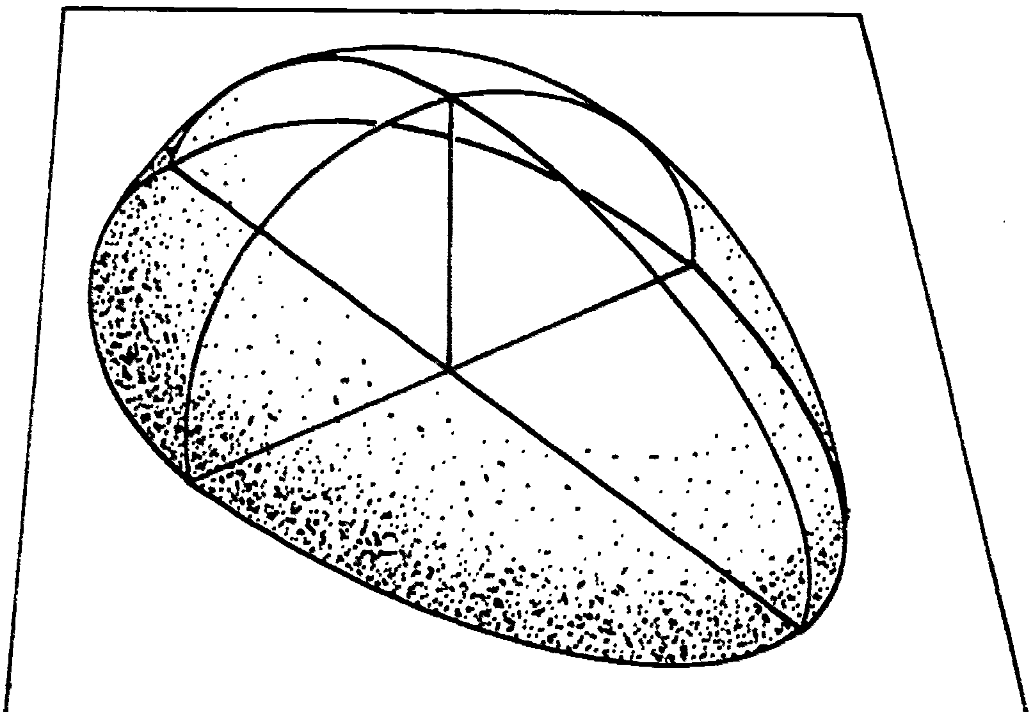
Everybody is familiar with the elements of plane geometry. A straight line is the shortest way between two points; parallel lines never meet; the sum of the three angles of a triangle equals two right angles, or 180 degrees, and so on and so forth. Also familiar is the geometry of figures drawn on the surface of a sphere, which obey somewhat different rules. The shortest route between two points on a sphere is called a "great circle"; this is the curve made by a cut through the points and the center of the sphere, splitting the sphere into equal halves. Two great circles always meet in two points; for instance, any two meridians of the earth always meet at the North and South poles. When segments of three great circles (for instance, one quarter of the earth's equator and the northern halves of two meridians) intersect to form a "spherical triangle," the three angles of 90 degrees add up to 270 degrees, or three right angles. The difference between this triangle and one in a plane derives from the fact that the sides of the former are drawn on a curved surface instead of on a flat one.

Now how do we know that the surface of a table is flat and that of the earth is spherical? All early civilizations imagined the earth as a flat disk, with mountains heaped upon it like food on the king's table. Not being able to go to the moon to look at the earth, men could not see its true shape. How, then, did Greek astronomers come to the conclusion that the earth was round? By observing that the North Star was higher in the sky in Greece than in Egypt. Thus it is evident that we can recognize that a sphere is round either by observing it from a distance or, if we stand on it, by observing objects far away.

Man also could, and did, discover that the earth was round in two



AN EGG has a curved surface which looks as if the surface of the large end belonged to one sphere and the surface of the small end to another. The middle has a different curvature.



HALF AN EGG, laid on a table and cut into vertical cross sections, will yield sections with concave curves downward. These curves look like portions of circles of different radii.

entirely different ways. One way was his circumnavigation of the earth. He found that while the surface of the earth had no "edge," no boundaries, its area was nevertheless limited. This is a most remarkable fact: the surface of the earth is boundless and yet it is finite. Obviously that situation rules out the possibility that the earth could be a plane. The surface of a plane is boundless and also infinite. (In common speech we consider these two words strictly synonymous — one of the many instances which prove that the sphericity of the world has not yet really taken hold of our consciousness.)

Thus mankind would have discovered that the earth is round even if it were constantly covered with a canopy of thick clouds. But suppose that he had somehow been prevented from exploring the whole planet. There is still another way in which he could have found out he was living on a globe, and that is by using the spherical geometry we have been talking about. If we look at a small triangle on the earth's surface, say one with sides about 30 feet long, it is indistinguishable from a flat triangle; the sum of its three angles exceeds 180 degrees by an amount so small that it cannot be measured. As we consider larger and larger triangles on the earth's spherical surface, however, their curvature will become more and more significant, and it will show up in the excess of the sum of their angles over 180 degrees. Thus by developing more and more precise methods of surveying and of making maps men eventually could prove the sphericity of the earth, and from their measurements they could find out the globe's radius. We shall return presently to this matter.

There are many types of surfaces besides those of a plane and a sphere. Consider an egg. It has a large end and a small end. A round piece of shell from the large end looks as if it were cut from a sphere; a round piece from the small end looks as if it belonged to a sphere with a smaller radius than the first. The piece from the small end looks more curved than that from the large end. Geometers define the curvature of a sphere as the inverse of its radius squared. So the smaller the radius, the larger the curvature, and *vice versa*.

If we were given a piece of shell from the middle zone of the egg, could we define its curvature? That is a little difficult, because such a piece cannot be identified with a portion of a simple sphere. The problem has been solved as follows. Suppose we lay the piece, which has the shape of a more or less elongated oval, on a table. It forms a rather flat dome. Any vertical cross section of that dome will be a curve concave downward. Every vertical cross section will look approximately like a portion of a circle, but not all will have the same radius. The section through the narrowest part of the base will have the smallest radius; the section through the elongated part, the largest. Let us call the first radius R_1

and the second R_2. Geometers then take a sort of average, and define the curvature of that small portion of eggshell as the inverse of the product R_1R_2. You can see that if the eggshell were a perfect sphere, we would be brought back to the previous definition.

On the basis of these definitions one finds that the curvature of a small piece of an eggshell changes as we travel on the surface of the egg. It would make no sense to talk about the curvature of the whole egg; we can only talk about the curvature of a small piece.

Consider next the surface of a saddle. A crosswise vertical section cut through a saddle forms a curve which is concave downward, whereas a lengthwise vertical cut forms a curve concave upward. This makes even a small piece of the surface of a saddle something radically different from a small piece of an eggshell. Geometers say that the eggshell has everywhere positive curvature, and the saddle has everywhere negative curvature. The curvature of a small portion of a saddle-shaped surface can again be defined as the inverse of the product R_1R_2, but this time it must be given a negative sign.

And here is still something else. Consider a doughnut. If you compare the inner half of the surface (facing the center of the hole in the doughnut) with the outer half, you will recognize that any small portion of the outer half has positive curvature, while any small portion of the inner half has negative curvature, as in the case of a saddle. Thus we must not think that the curvature need be positive or negative all over a given surface; as we travel from point to point on a surface the curvature not only can become greater or smaller, it can also change its sign.

Remember that we are engaged in taking a bird's-eye view of what was known about the curvature of surfaces before Riemann's time. What we have seen so far had been recognized in the 18th century by Leonard Euler, a Swiss mathematician of considerable imagination and output, and had been developed by a group of French geometers at the newly founded Ecole Polytechnique. Then in 1827 Gauss, Riemann's senior examiner, added much generality and precision to the topic. He published a memoir on curved surfaces which is so jewel-perfect that one can still use it today in a college course.

Gauss started from the fact that geographers specify the location of a city on the globe by giving its longitude and latitude. They draw meridians of longitude (such as the one which unites all the points on the globe 85 degrees west of the north-and-south great circle through Greenwich) and also parallels of latitude. We may speak of the "family" of meridians and the "family" of parallels. In order to specify the location of a point on any mathematically given surface, Gauss imagined that we draw on that surface two families of curves, called p-curves and q-curves. We take

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suitable precautions so that any point on the surface will be pinpointed if we specify its p-coordinate and its q-coordinate.

Gauss's great insight was this. On an absolutely flat surface, if we travel three miles in one direction, then turn left and travel four miles in the perpendicular direction, we know from Pythagoras' theorem that we are at a point five miles from home. But Gauss reasoned that on a curved surface, whether egg, or saddle or what have you, the distance will be different. To begin with, the p-curves and q-curves will not intersect everywhere at right angles, and this adds a third term to the sum of the two squares in the Pythagorean equation $a^2 + b^2 = c^2$. Moreover, if we visualize the two families of curves as a kind of fish net drawn tight all over the surface, the angles and sides of the small meshes will change slowly as we travel from one region of the surface to another where the curvature is different.

Gauss expressed his reasoning in a famous mathematical equation. One p-curve and one q-curve pass through a given point M on a curved surface. The "quasi longitude" p and the "quasi latitude" q of point M have specific numerical values. We wish to move from point M to a neighboring point P on the surface. We first increase the value of p by a small quantity, letting q remain the same. Gauss used dp as the symbol for an arbitrarily small increase of p. We thus get to a point N, of longitude $p + dp$ and latitude q. We next increase the value of q by a small quantity, dq, letting $p + dp$ remain the same. We thus reach a point P, of longitude $p + dp$ and latitude $q + dq$. We wish to know the distance from point M to point P. Since this distance is arbitrarily small, Gauss used for it the symbol ds. In Gauss's notation, the square of the distance ds will be expressed by the sum of three terms:

$$ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2$$

This equation is one of the high points in the whole of mathematics and physics — a mountain-top where we should exclaim in awe, like Faust suddenly perceiving the symbol of the macrocosm: "Was he a god, whoever wrote these signs?" It needed only two steps, one taken by Riemann and the other by Einstein, to carry us from Gauss's equation into the land of general relativity.

At any point M on our arbitrary surface, this equation is not different from a Euclidean theorem about the square of the third side, ds, of any triangle, the first two being dp, dq. That is because in the immediate neighborhood of a point the surface is very nearly a plane. But here is the novelty: Gauss introduced the functions E, F and G, whose numerical values change continuously as we move from point to point on the surface. Gauss saw that each of the quantities E, F, G was a function of the two arbitrary quantities p and q, the quasi longitude and the quasi latitude of point M. On a plane we can draw p-lines and q-lines dividing the plane into small equal squares, as on a chessboard; we have then $ds^2 = dp^2 + dq^2$, so that E is constantly equal to 1, F to zero and G to 1 all over the plane. But on a curved surface E, F and G vary in a way which expresses, in an abstract but precise manner, just those variations in the curvature of a surface that make every point different from every other.

Gauss now proved this remarkable theorem: that the curvature of the surface at any point can be found as soon as one knows the values of E, F and G at the point, and how they vary in its immediate neighborhood. Why is this theorem so remarkable? Because if we return to our fictional humanity living on some beclouded globe, not a spherical one this time but of arbitrary shape, the surveyors of any particular nation on that globe, knowing the theorem, could obtain all the information about E, F and G without seeing the stars and without going to the moon. Thus from measurements taken on the surface itself they would be able to calculate the curvature of their globe at various points and to find out whether the surface of their country was curved like a portion of an egg, saddle or doughnut, as the case might be.

Now of all the ridiculous and useless puzzles scientists like to solve, this one, you may think, surely takes the prize. Why should mathematicians find it important to describe the behavior of imaginary people in a nonexistent world? For a very good reason: *These people are ourselves.* Only it takes some little explanation to make you realize I have been talking about you and me.

Let us imagine small bits of paper of various irregular shapes on a large, smooth sphere. These bits of paper are alive and moving: they are the people of that world, only their bodies are not volumes enclosed by surfaces but surfaces enclosed by curves. These people, having absolutely flat bodies without thickness, can form no conception of the space above or below them. They are themselves only portions of surfaces, two-dimensional beings. Their senses are adapted to give them information about the surroundings in their two-dimensional world. But they have no experience whatsoever of anything outside that world; so they cannot conceive of a third dimension.

However, they are intelligent; they have discovered mathematics and physics. Their geometry consists of two parts — line geometry and plane geometry. In physics they illustrate problems in one variable by diagrams on a line; problems in two variables, by surface diagrams. Problems in three, four or more variables they solve by algebra: "It's too bad," they say, "that for these we can't have the help of diagrams."

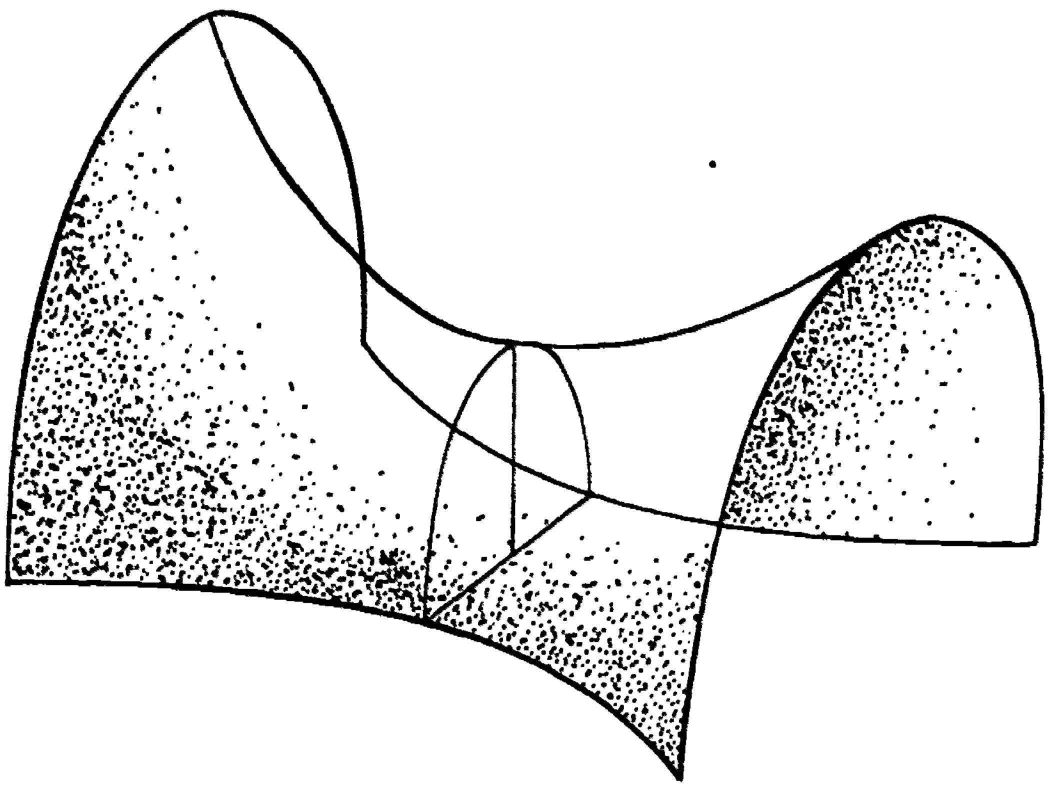
In the first half of the 19th century (*their* 19th century) an idea dawned upon several of them. "We cannot," they said, "imagine a third dimension, but we do handle physical problems in three variables, x, y, z. Why couldn't we *talk* about a space of three dimensions? Even if we cannot visualize it, it might be helpful to be able to talk about points, lines and areas located in that space. Maybe something might come of it: anyway, there's no harm in trying." And so they tried.

We need not carry this fable any farther; its meaning is clear enough. We are just like these people, only our bodies have three dimensions and are moving about in a three-dimensional world. Neither you nor I can visualize a fourth dimension; yet we handle problems about a particle moving in space, and this is a problem in four variables: x, y, z for space and t for time. We also handle problems about electromagnetic fields. Well, the electric field vector E at any point (x, y, z) has three projections, E_x, E_y, E_z, and it changes in space and time; that makes seven variables. Add three more for its twin brother, the magnetic field B, and we have 10. It looks as if the mathematical physicist could well use spaces of four or 10 or any number of dimensions.

Riemann in his dissertation assumed at the outset a space of an arbitrary number of dimensions. Now a lesser geometer would have found it very straightforward to define the distance of two neighboring points in that space. Don't we know from Pythagoras' theorem that in a plane the square of that distance, ds^2, is equal to the sum of two squares: $ds^2 = dx^2 + dy^2$? Well then obviously in an n-dimensional space ds^2 must be the sum of n squares, the sum of all the terms similar to dx^2 which we can find. A very convenient shorthand for the expression "the sum of all the terms similar to" is the Greek capital Σ. Thus a simple-minded geometer would have written $ds^2 = \Sigma\, dx^2$. But Riemann saw farther than that. He had given much thought to the 1827 memoir of his master Gauss. He reasoned that, if we assumed that $ds^2 = \Sigma\, dx^2$, we were beaten at the outset. For Pythagoras' theorem is valid only in a plane, divided into equal little squares like a chessboard. Actually what we need to generalize is Gauss's equation, which works for any curved surface whatsoever, including a plane as a very special case. Gauss had added two things to Pythagoras' formula: (1) to the squares of dp and dq he had added the product $dp\, dq$ of these two quantities; (2) he had multiplied each of these three terms by a coefficient of its own, and assumed that these coefficients E, F, G varied from point to point over the surface.

Let us do the same thing, then, for a "supersurface" of three dimensions, whatever that may be. We shall stretch over this supersurface three families of surfaces p, q, r or, as they are more conveniently designated, x_1, x_2, x_3. The square of the distance between two neighboring points, ds^2, should be built not only from the squares of dx_1, dx_2, dx_3, but also from their products two by two, and there are three such products: dx_2dx_3, dx_3dx_1 and dx_1dx_2. This makes a total of six terms, and we must give them six coefficients. Let us represent these coefficients by the letter g, with suitable subscripts. We must then write: $ds^2 = g_{11}dx_1^2 + g_{22}dx_2^2 + g_{33}dx_3^2 + 2g_{23}dx_2dx_3 + 2g_{31}dx_3dx_1 + 2g_{12}dx_1dx_2$. (The factor 2 is not indispensable, but it is esthetically satisfying to the algebraist, and Gauss had taken a fit when a young Berlin professor, Dirichlet, had committed the *faux pas* of writing a memoir which dispensed with the factor 2.) This, then, is the correct form of the ds^2 for a supersurface of three dimensions, and the six coefficients will in general vary from point to point over the supersurface.

Riemannn, as we have said, assumed at the outset that he had *n* variables to deal with, not a specific number such as three or four. He needed a name for the kind of geometrical objects he was thinking about. He noticed two things. First, a particle is free (in theory) to move smoothly and continuously from one point of a line or curve to another; it may



A SADDLE cut into lengthwise cross sections forms curves upward, while crosswise sections curve downward with shorter radii. A saddle is described as having negative curvature.

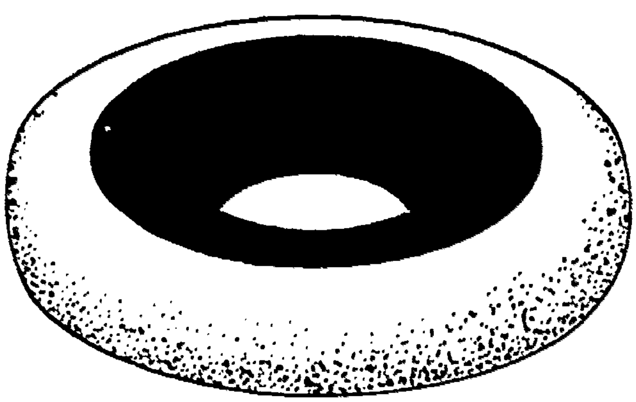
also move continuously from one point to another on a surface or in space. Second, while studying plane geometry we think of nothing but figures drawn on a plane; that plane is for the time being our whole "universe of discourse," as logicians say. Yet the next year, as we study solid geometry, we imagine planes of any orientation in space. Any one of these planes might well be *the* plane of plane geometry which was last year's universe of discourse. It makes no difference in the geometry of a plane whether this plane exists all by itself or whether it is "embedded," as we now say, in three-dimensional space.

Putting these remarks together, Riemann coined the name "continuum" for any geometrical object, of any number of dimensions, upon which a point can continuously roam about. A straight line, for instance, is a continuum in one dimension — and it makes no difference to the geometry of points and segments on that line whether this one-dimensional continuum exists all by itself or is embedded in a plane, in three-dimensional space or for that matter in a space of any number of dimensions. The surface of a sphere or of a saddle is, as we have seen, a two-dimensional continuum; again it makes no difference whether we consider it by itself or embedded in a space of any number of dimensions.

Now our space is a three-dimensional continuum. And we are bound to add that geometry in our space will be the same whether we consider that space by itself or assume it is embedded in a space of four, five or any number of dimensions. We cannot visualize what this means. Just the same, we might follow up this trail and see where logic leads us.

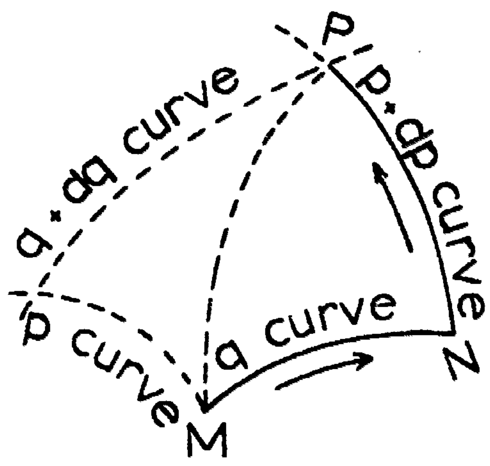
Such must have been young Riemann's thoughts about the year 1850. We must now try to say in a few words how far he progressed from there, and what, mainly, his dissertation of 1854 contained.

At a first reading, the outstanding result of Riemann's efforts seems

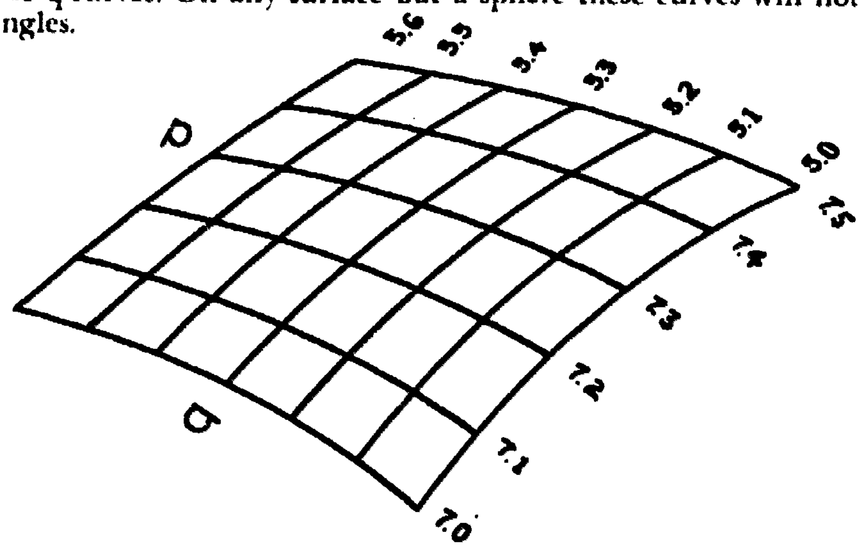


A DOUGHNUT'S SURFACE shows positive curvature in its outer half, while the inner half has negative curvature (*black*).

to be that he succeeded in defining the curvature of a continuum of more than two dimensions. A two-dimensional continuum is a surface, and we have seen that its curvature is defined, for a small region surrounding any point of the surface, by a single number positive on an egg-shaped surface, negative on a saddle-shaped surface. If the curvature is zero at every point, the surface is a plane, and *vice versa*. Riemann showed that the concept of curvature can be generalized for the case of a continuum of n dimensions. Only it will not be a single number any



LOCATION of a point on any mathematically given surface may be specified by giving one coordinate from the family of p-curves and one from the intersecting family of q-curves. On any surface but a sphere these curves will not intersect at right angles.



DISTANCE from one point P to a point M on a surface of any curvature cannot be determined by the Pythagorean rule. Gauss defined it as a function of the intersecting coordinates locating points and curvature varying on the surface from point to point.

more; a set of three numbers will be needed to define the curvature of a continuum of three dimensions, a set of six numbers for one of four dimensions, and so forth. Riemann only stated these results and made them seem mathematically plausible; their proof and elaboration would have filled a long memoir or occupied several weeks of lectures.

These considerations seem purely abstract—a completely vacuous game of mathematics running wild. However, Riemann's main object in his dissertation was to convince us that he was talking not about abtsract mathematical concepts but about a question of physics which could be settled by the experimental method.

Let us return to those perfectly flat beings that live on a huge surface. Gauss's "remarkable theorem" proves that the two-dimensional inhabitants of this two-dimensional universe, provided they understood enough mathematics, could find the curvature of any small region of their universe. How could these people conceive of a curved surface, if they could not visualize a space of three dimensions? The answer is that such is precisely the power of mathematics. These people would be familiar with the concept of a curved road, contrasting with a "straight" road which would be the shortest route between two points. If then some Riemann among them had generalized this notion, *in a purely algebraic way*, into a theory of the curvature of a continuum of n dimensions, their surveyors would be able to calculate from a formula given by Riemann a certain number which, they would find, would change slightly from country to country. Thus they would have measured the curvature of their two-dimensional universe without being able in any way to visualize what that could be.

Such, of course, is exactly our situation regarding the curvature of our own universe, and we must return to Riemann's work to form some idea of how he came to define it.

Riemann suggested that if all the numbers which defined the curvature of an n-dimensional space were zero, this space should be called flat, for that is what we call a surface whose curvature is zero. Now if we divide a three-dimensional space into equal little cubes, as a chessboard is divided into equal little squares, then ds^2 is simply the sum $dx^2 + dy^2 + dz^2$, with dx, dy, dz representing the three sides of each little cube. That space is a "flat" space, just as a plane is a flat surface. In other words, what our intuition tells us is that space is flat—in the sense given to that word by Riemann.

Is it really so? That the small portion of space in our neighborhood should appear flat is only to be expected. It may well be that space is actually flat, not only in our vicinity but away into the realm of the farthest nebulae. On the other hand, it is equally possible that space is ever so slightly curved. How could we ever find out? Riemann's answer was: *from experience*. That is the revolutionary message which, very quietly but very firmly, he brought to the scientific world.

Euclid and Kant had unconsciously accepted the intuitive notion of space as flat. Riemann declared that this proposition should not be asserted without proof, as self-evident; it was only a hypothesis, subject to test by experiment. To start with we could make three hypotheses about our space: that it had constant positive curvature, or constant negative curvature or no curvature at all (i.e., that it was flat, or Euclidean, as we now say). Which of these hypotheses was correct was for astronomers and physicists to find out. Such was the meaning of Riemann's cryptic title, "On the Hypotheses That Are the Foundations of Geometry," which had, how very rightly, aroused the curiosity of Gauss.

There are many other important things in this dissertation of Riemann's, such as a very clear-sighted appreciation of the possibility that we may have to adopt eventually a quantum theory of space — something our physicists are just now rather gingerly trying out. But the point we have presented here — the appeal to experiment in order to find out a possible curvature of space — is, we believe, the most important one.

Riemann wisely made no attempt to suggest what specific experiments should be made. Looking back from the vantage point of our post-Einsteinian knowledge, we realize they were very difficult to discover. One might have expected them to lie in the domain of classical astronomy, of measurement of angles between stars, but that doesn't cut deep enough. Einstein showed that gravitation had a great deal to do with the matter and that Riemann's provisional hypothesis of a space of constant curvature had to be abandoned in favor of local variations (e.g., the curvature in the neighborhood of the sun or of Sirius was greater than in empty interstellar space). He also showed that time had to be brought in; in other words, a four-dimensional space-time was what had to be investigated experimentally. And thus it came about that in the three experimental checks on Einstein's theory obtained in 1920, space, time and gravitation were seen to be indissolubly mixed.

Riemann's contention that the geometry of the universe was just a chapter of physics, to be advanced like any other by the close cooperation of theory and experiment, was thereby fully justified. So also was Riemann's faith in his master, Gauss. The more we gaze upon Riemann's and Einstein's truly gigantic pyramids of thought, the more we admire how much was invisibly contained in the short, unassuming formula written by Gauss in 1827.

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—W.L.S.

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